Discrete Applied Mathematics 159 (2011) 1488–1507

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Incoherent majorities: The McGarvey problem in judgement aggregation

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ARTICLE INFO

Article history:
Received 10 May 2010
Received in revised form 19 March 2011
Accepted 24 March 2011
Available online 27 April 2011

Keywords:
Judgement aggregation
Majority voting
McGarvey
Stearns
0/1 polytope

ABSTRACT

Judgement aggregation is a model of social choice where the space of social alternatives is the set of consistent truth-valuations ('judgements') on a family of logically interconnected propositions. It is well known that propositionwise majority voting can yield logically inconsistent judgements. We show that, for a variety of spaces, propositionwise majority voting can yield *any* possible judgement. By considering the geometry of sub-polytopes of the Hamming cube, we also estimate the number of voters required to achieve all possible judgements. These results generalize the classic results of McGarvey (1953) [13] and Stearns (1959) [22].

Let $\mathcal{K}$ be a finite set of propositions or 'properties'. An element $\mathbf{x} = (x_k)_{k\in\mathcal{K}} \in \{\pm 1\}^{\mathcal{K}}$ is called a *judgement*, and interpreted as an assignment of a truth value of 'true' $(+1)$ or 'false' (-1) to each proposition. Not all judgements are feasible, because there will be logical constraints between the propositions (determined by the structure of the underlying decision problem faced by the voters). Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be the set of 'admissible' judgements — we refer to $\mathcal{X}$ as a *property space*. An *anonymous profile* is a probability measure on $\mathcal{X}$ — that is, a function $\mu : \mathcal{X} \longrightarrow [0, 1]$ such that $\sum_{\mathbf{x}\in\mathcal{X}} \mu(\mathbf{x}) = 1$. (Interpretation: for all $\mathbf{x} \in \mathcal{X}$, $\mu(\mathbf{x})$ is the proportion of the voters who hold the judgement $\mathbf{x}$). Let $\Delta(\mathcal{X})$ be the set of all anonymous profiles. *Judgement aggregation* is the problem of converting a profile $\mu \in \Delta(\mathcal{X})$ into the element $\mathbf{x} \in \mathcal{X}$ which best represents the 'collective will' of the voters. This problem (with different terminology) was originally posed by Guilbaud [7], and later investigated by Wilson [25], Rubinstein and Fishburn [20], and Barthelémy and Janowitz [3]. Since the work of List and Pettit [11], there has been an explosion of interest in this area; see [12] for a recent survey of judgement aggregation research.

For example, let $\mathcal{A}$ be a finite set of social alternatives. A *tournament* on $\mathcal{A}$ is a complete antisymmetric relation "$\prec$" over $\mathcal{A}$. A *preference order* is a transitive tournament (i.e. a linear ordering) on $\mathcal{A}$. Let $\mathcal{K} \subset \mathcal{A} \times \mathcal{A}$ contain exactly one of the pairs (a, b) or (b, a) for each distinct $a, b \in \mathcal{A}$. Any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ represents a tournament "$\prec$", where $a \prec b$ iff $x_{a,b} = 1$. Every tournament on $\mathcal{A}$ corresponds to a unique element of $\{\pm 1\}^{\mathcal{K}}$. Let $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$ denote the subset of all elements of $\{\pm 1\}^{\mathcal{K}}$ which correspond to preference orders. Thus, a profile $\mu \in \Delta(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}})$ represents a group of voters who each assert some preference order over $\mathcal{A}$. In this case, the goal of judgement aggregation is to distill μ into some 'collective' preference order on $\mathcal{A}$ — this is the familiar Arrovian model of preference aggregation.

Propositionwise majority vote is defined as follows. For any $\mu \in \Delta(\mathcal{X})$, any $k \in \mathcal{K}$, let

$$\widetilde{\mu}_k := \sum_{\mathbf{x}\in\mathcal{X}} \mu(\mathbf{x}) \cdot x_k \tag{1}$$

be the μ-expected value of coordinate x_k. Thus, $\widetilde{\mu}_k > 0$ if and only if a strict majority of voters assert '$x_k = 1$'; whereas $\widetilde{\mu}_k < 0$ if and only if a strict majority of voters assert '$x_k = -1$'. Let $\Delta^*(\mathcal{X}) := \{\mu \in \Delta(\mathcal{X}); \widetilde{\mu}_k \neq 0, \forall k \in \mathcal{K}\}$ be the set of

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2011.03.014

anonymous profiles where there is a strict majority supporting either $+1$ or -1 in each coordinate.[1] For any $\mu \in \Delta^*(\mathcal{X})$, define $\mathrm{maj}(\mu) \in \{\pm 1\}^{\mathcal{K}}$ as follows:

$$\text{for all } k \in \mathcal{K}, \quad \mathrm{maj}_k(\mu) := \begin{cases} 1 & \text{if } \widetilde{\mu}_k > 0; \\ -1 & \text{if } \widetilde{\mu}_k < 0. \end{cases} \tag{2}$$

Unfortunately, it is quite common to find that $\mathrm{maj}(\mu) \notin \mathcal{X}$ —the 'majority will' can be inconsistent with the underlying logical constraints faced by the voters. (In the case of aggregation over $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$ with $|\mathcal{A}| \geq 3$, this problem was first observed by Condorcet [4].) Let $\mathrm{maj}(\mathcal{X}) := \{\mathrm{maj}(\mu);\ \mu \in \Delta^*(\mathcal{X})\}$; this describes the set of all majoritarian voting patterns that can result from some possible profile of judgements. Following McGarvey [13], we think of $\mathrm{maj}(\mathcal{X}) \setminus \mathcal{X}$ as the range of possible 'voting paradoxes' which can occur under propositionwise majority vote.

Clearly $\mathcal{X} \subseteq \mathrm{maj}(\mathcal{X})$. We say that $\mathcal{X}$ is *majority consistent* if $\mathrm{maj}(\mathcal{X}) = \mathcal{X}$. This occurs only when $\mathcal{X}$ satisfies a strong combinatorial/geometric condition, as we new explain. For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathcal{X}$, we define $\mathrm{med}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \mathrm{maj}(\mu)$, where $\mu \in \Delta^*(\mathcal{X})$ is defined by $\mu(\mathbf{x}_j) = \frac{1}{3}$ for $j = 1, 2, 3$; this defines a ternary operator on $\{\pm 1\}^{\mathcal{K}}$, called the *median operator*. Let $\mathrm{med}^1(\mathcal{X}) := \{\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z});\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}\}$. For all $n \in \mathbb{N}$, we inductively define $\mathrm{med}^{n+1}(\mathcal{X}) := \{\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z});\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{med}^n(\mathcal{X})\}$. This yields an ascending chain $\mathcal{X} \subseteq \mathrm{med}^1(\mathcal{X}) \subseteq \mathrm{med}^2(\mathcal{X}) \subseteq \cdots$. Let $\mathrm{med}^\infty(\mathcal{X}) := \bigcup_{n=1}^\infty \mathrm{med}^n(\mathcal{X})$ be the *median closure* of $\mathcal{X}$. We say that $\mathcal{X}$ is a *median space* if $\mathrm{med}^1(\mathcal{X}) = \mathcal{X}$ (equivalently, $\mathrm{med}^\infty(\mathcal{X}) = \mathcal{X}$). At the opposite extreme, $\mathcal{X}$ is *median-saturating* if $\mathrm{med}^\infty(\mathcal{X}) = \{\pm 1\}^{\mathcal{K}}$. For any $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$, we have

$$\mathcal{X} \subseteq \mathrm{med}^1(\mathcal{X}) \subseteq \mathrm{maj}(\mathcal{X}) \subseteq \mathrm{med}^\infty(\mathcal{X}). \tag{3}$$

The first two inclusions are obvious by definition. The last inclusion is due to Nehring and Puppe [17]; see also [18].[2] It follows that $\mathcal{X}$ is majority consistent if and only if $\mathcal{X}$ is a median space. If $\mathcal{X}$ is *not* a median space, then Eq. (3) is useful because it is relatively easy to compute $\mathrm{med}^\infty(\mathcal{X})$, as we now explain.

Let $\mathcal{J} \subseteq \mathcal{K}$ and let $\mathbf{w} \in \{\pm 1\}^{\mathcal{J}}$; we say that $\mathbf{w}$ is a *word* (or sometimes, $\mathcal{J}$*-word*) and call $\mathcal{J}$ the *support* of $\mathbf{w}$, denoted $\mathrm{supp}(\mathbf{w})$. If $\mathcal{I} \subseteq \mathcal{J}$ and $\mathbf{v} \in \{\pm 1\}^{\mathcal{I}}$, then we write $\mathbf{v} \sqsubseteq \mathbf{w}$ if $v_i = w_i$ for all $i \in \mathcal{I}$. We define $|\mathbf{w}| := |\mathcal{J}|$. We say $\mathbf{w}$ is an $\mathcal{X}$*-forbidden word* if, for all $\mathbf{x} \in \mathcal{X}$, we have $\mathbf{w} \not\sqsubseteq \mathbf{x}$. Let $\mathcal{W}_2(\mathcal{X})$ be the set of all $\mathcal{X}$-forbidden words of length 2. We obtain the following proposition.

Proposition 1.1. *Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$.*

(a) $\mathrm{med}^\infty(\mathcal{X}) := \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}};\ \mathbf{w} \not\sqsubset \mathbf{x},\ \forall \mathbf{w} \in \mathcal{W}_2(\mathcal{X})\}$.
(b) *In particular, $\mathcal{X}$ is median-saturating if and only if $\mathcal{W}_2(\mathcal{X}) = \emptyset$.*

(The proof of this and all other results are in the Appendix at the end of the paper.)

Example 1.2. Let $\mathcal{N}$ be a set and let $\mathcal{K} := \{(n, m) \in \mathcal{N} \times \mathcal{N};\ n \neq m\}$; then any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ represents a binary relation "$\preceq$" on $\mathcal{N}$ such that $n \preceq m$ if and only if $x_{n,m} = 1$. Let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ be any space of *complete* binary relations. Then $\mathcal{W}_2(\mathcal{X}) \neq \emptyset$, because for any $\mathbf{x} \in \mathcal{X}$ and $(n, m) \in \mathcal{K}$, we cannot have both $x_{n,m} = -1$ and $x_{m,n} = -1$ (by completeness). Thus, $\mathrm{med}^\infty(\mathcal{X}) \neq \{\pm 1\}^{\mathcal{K}}$. □

Given a property space $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$, Proposition 1.1 and Eq. (3) raise the question: Is

$$\mathrm{maj}(\mathcal{X}) = \mathrm{med}^\infty(\mathcal{X})?. \tag{4}$$

Clearly, if $\mathcal{X}$ is a median space, then Eq. (3) implies that $\mathrm{maj}(\mathcal{X}) = \mathrm{med}^\infty(\mathcal{X})$. At the other end of the spectrum, McGarvey [13] showed that $\mathrm{maj}(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) = \{\pm 1\}^{\mathcal{K}}$ whenever $|\mathcal{A}| \geq 3$; this automatically implies that $\mathrm{maj}(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) = \mathrm{med}^\infty(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}})$. Hollard and le Breton [8] and Vidu [24] obtained McGarvey-type results for separable preferences (see Example 3.4 below), while Shelah [21] has recently extended McGarvey's result to the case when $\mathcal{X}$ represents any collection of tournaments on $\mathcal{A}$ which is invariant under vertex permutations (see Proposition 3.6).[3]

Question (4) appears to be difficult to answer in full generality. We will thus focus on the special case when Eq. (4) holds and $\mathcal{X}$ is median-saturating — in other words, we ask when $\mathrm{maj}(\mathcal{X}) = \{\pm 1\}^{\mathcal{K}}$. In this case, we say that $\mathcal{X}$ is *McGarvey*.

If $\mathcal{X}$ is McGarvey, then every conceivable 'voting paradox' can be obtained through propositionwise majority voting on $\mathcal{X}$. The McGarvey property is also useful in establishing other results about $\mathcal{X}$. For example, Nehring et al. [16] consider other judgement aggregation rules on $\mathcal{X}$ based on 'Condorcet efficiency' (a generalization of the 'Condorcet principle' of preference aggregation). The McGarvey property of certain property spaces is part of the reason that Condorcet efficient judgement aggregation can be quite indeterminate on those spaces.

[1] Usually, judgement aggregation is considered on *all* of $\Delta(\mathcal{X})$. However, we will confine our attention to profiles in $\Delta^*(\mathcal{X})$ for expositional simplicity. (If the set of voters is large (respectively odd), then a profile in $\Delta(\mathcal{X}) \setminus \Delta^*(\mathcal{X})$ is highly unlikely (respectively impossible) anyways.)

[2] The close relationship between the median operator and majoritarian consensus on median graphs and median lattices had earlier been explored by Guilbaud [7], Barthélémy and Janowitz [3], McMorris et al. [14] and others.

[3] Shelah [21] also proves other, more general results about $\mathrm{maj}(\mathcal{X})$ when $\mathcal{X}$ represents a symmetric set of tournaments.

The central question of this paper is: *What property spaces are McGarvey?* Let $\mathrm{conv}(\mathcal{X})$ denote the convex hull of $\mathcal{X}$ (seen as a subset of $\mathbb{R}^{\mathcal{K}}$), and let $\mathrm{int}[\mathrm{conv}(\mathcal{X})]$ denote its topological interior. Let $\mathbf{0} := (0, 0, \dots, 0) \in \mathbb{R}^{\mathcal{K}}$. For any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, the *open orthant* of $\mathbf{x}$ is the open set $\mathcal{O}_{\mathbf{x}} := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}};\ \mathrm{sign}(r_k) = x_k,\ \forall k \in \mathcal{K}\}$. Most of the results in this paper are based on the following characterization of McGarvey spaces.

Theorem 1.3. *Let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$. Then*

(a) $\mathrm{maj}(\mathcal{X}) = \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}};\ \mathcal{O}_{\mathbf{x}} \cap \mathrm{conv}(\mathcal{X}) \neq \emptyset\}$.
(b) *The following are equivalent:*
  (b1) *$\mathcal{X}$ is McGarvey;*
  (b2) $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{X})]$*;*
  (b3) *For every nonzero $\mathbf{z} \in \mathbb{R}^{\mathcal{K}}$, there exists $\mathbf{x} \in \mathcal{X}$ with $\mathbf{z} \bullet \mathbf{x} > 0$.*
  (b4) $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$*, and $\mathbf{0}$ is a strictly positive convex combination of elements of $\mathcal{X}$.*
  (b5) $\mathrm{cone}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$.

Conditions (b2) and (b5) locate the McGarvey problem in the theory of convex polytopes. In applications, falsifying (b3) is often the easiest way to show that $\mathcal{X}$ is *not* McGarvey, while (b4) is a handy method to show that $\mathcal{X}$ *is* McGarvey (in practice, most judgement aggregation problems satisfy the hypothesis $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$). Condition (b5) implies that, not only can we realize any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ by a majority vote, but further, we can realize any given ratio of supermajorities supporting the various coordinates $\mathbf{x}$; this is useful in the study of certain 'supermajoritarian efficient' judgement aggregation rules [15].

The rest of this paper is organized as follows. In Section 2, we ask how small $\mathcal{X}$ can be while still being McGarvey, or how large it can be without being McGarvey. In Section 3, we characterize the McGarvey property for judgement aggregation spaces with many symmetries; this includes spaces of preference relations, equivalence relations, and connected graphs, and also leads to a simpler proofs of the results of Hollard and le Breton [8], Vidu [24], and Shelah [21]. In Sections 4–6 we consider the McGarvey problem for comprehensive spaces, truth-functional aggregation spaces, and convexity spaces, respectively. Finally, in Section 7, we consider a problem originally studied by Stearns [22]: How *many* voters are required to realize the McGarvey property of a space $\mathcal{X}$? We show that several important families of aggregation spaces only require around $2K$ voters. However, using a result of Alon and Vũ [2], we also show that the required number of voters can be extremely large for some McGarvey spaces.

Throughout this paper, we make the following assumption without loss of generality: for all $k \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ such that $x_k \neq x'_k$ (otherwise one can just remove k from $\mathcal{K}$). We will also assume $|\mathcal{K}| \geq 3$ (otherwise the McGarvey problem is trivial).

2. Minimal McGarvey spaces and maximal non-McGarvey spaces

If $\mathcal{X} \subseteq \mathcal{Y} \subseteq \{\pm 1\}^{\mathcal{K}}$, and $\mathcal{X}$ is McGarvey, then clearly $\mathcal{Y}$ is also McGarvey. It is therefore interesting to study 'minimal' McGarvey spaces. We say that $\mathcal{X}$ is *minimal McGarvey* if $\mathcal{X}$ is McGarvey, but no proper subset of $\mathcal{X}$ is McGarvey. For the next result and the rest of the paper, we define $K := |\mathcal{K}|$.

Proposition 2.1. **(a)** *Suppose $K \geq 3$. Then* $\min\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *is McGarvey*$\} = K + 1$.
(b) $\max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *is minimal McGarvey*$\} = 2K$.

Example 2.2. Suppose $K \geq 3$. For all $j \in \mathcal{K}$, define $\boldsymbol{\chi}^j \in \{\pm 1\}^{\mathcal{K}}$ by $\chi^j_j := 1$, while $\chi^j_k := -1$ for all $k \in \mathcal{K} \setminus \{j\}$. Define $\mathcal{X} := \{\pm \boldsymbol{\chi}^j\}_{j \in \mathcal{K}}$. Then $|\mathcal{X}| = 2K$. In the Appendix, we show that $\mathcal{X}$ is a minimal McGarvey space. In particular, if $K = 3$, then $\mathcal{X} = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, 1), (-1, 1, -1), (1, -1, -1)\}$ is a minimal McGarvey set with six elements. Let $\mathcal{A} := \{a, b, c\}$ and identify $\mathcal{K}$ with the set $\{(a, b), (b, c), (c, a)\}$; then $\mathcal{X} = \mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$. □

By comparison, Carathéodory's theorem says that if $\mathcal{Y} \subset \{\pm 1\}^{\mathcal{K}}$ is a minimal set with $\mathbf{0} \in \mathrm{conv}(\mathcal{Y})$, then $2 \leq |\mathcal{Y}| \leq K + 1$. The requirement that $\mathbf{0}$ be in the *interior* of $\mathrm{conv}(\mathcal{Y})$ instead entails $K + 1 \leq |\mathcal{Y}| \leq 2K$; this shows that the interiority condition is quite substantive.

For further comparison, we say that $\mathcal{X}$ is *minimal median-saturating* if $\mathcal{X}$ is median-saturating, but no proper subset of $\mathcal{X}$ is median-saturating.

Proposition 2.3. *For any $K \in \mathbb{N}$, let* $m(K) := \min\{|\mathcal{X}|;\ \mathcal{X} \subseteq \{\pm 1\}^K$ *is median-saturating*$\}$.

(a) *(Kleitman et al. [10,9])* $m(K) = \min\left\{M \in \mathbb{N};\ K \leq \binom{M-1}{\lfloor \frac{M}{2} \rfloor - 1}\right\}$.
(b) $\lceil \log_2(K) \rceil + 1 \leq m(K) \leq 2\lceil \log_2(K) \rceil + 2$ *for all* $K \in \mathbb{N}$.
(c) $m(K) = \log_2(K) + \frac{1}{2}\log_2(\log_2(K)) + \mathcal{O}(1)$ *as* $K \to \infty$.
(d) *If $K \geq 4$, then* $K(K-1)/2 \leq \max\{|\mathcal{X}|;\ \mathcal{X} \subseteq \{\pm 1\}^K$ *is minimal median-saturating*$\} \leq 2K(K-1)$.

A comparison of Propositions 2.1 and 2.3 indicates how median saturation is substantially weaker than the McGarvey property.

Proposition 2.4. (a) $\max\{|\mathcal{X}|;\ \mathcal{X}\subset\{\pm1\}^{\mathcal{K}}$ *is not McGarvey*$\} = \frac{3}{4}2^K$.
(b) $\max\{|\mathcal{X}|;\ \mathcal{X}\subset\{\pm1\}^{\mathcal{K}}$ *is not median-saturating*$\} = \frac{3}{4}2^K$.

Example 2.5. Let $\mathcal{K}=\{1,2,\ldots,K\}$ and let $\mathcal{X}:=\{\mathbf{x}\in\{\pm1\}^{\mathcal{K}};\ (x_1,x_2)\neq(-1,-1)\}$. Then $\mathcal{X}$ is a median space (hence, neither McGarvey nor median-saturating) but $|\mathcal{X}|=\frac{3}{4}2^K$. (Also, note that $\mathbf{0}\in\operatorname{conv}(\mathcal{X})$ and $\operatorname{int}[\operatorname{conv}(\mathcal{X})]\neq\emptyset$; this shows that the McGarvey property is stronger than the conjunction of these two conditions.) □

Propositions 2.1 and 2.4 show that the McGarvey property places only very loose constraints on the cardinality of $\mathcal{X}$. Much more important is how 'dispersed' $\mathcal{X}$ is as a subset of $\{\pm1\}^{\mathcal{K}}$.

3. Symmetric property spaces

For any $\mathcal{X}\subset\mathbb{R}^{\mathcal{K}}$, the *symmetry group* of $\mathcal{X}$ is the set $\Gamma_{\mathcal{X}}$ of all invertible linear transformations $\gamma:\mathbb{R}^{\mathcal{K}}\longrightarrow\mathbb{R}^{\mathcal{K}}$ such that $\gamma(\mathcal{X})=\mathcal{X}$. Let $\operatorname{Fix}(\Gamma_{\mathcal{X}}):=\{\mathbf{r}\in\mathbb{R}^{\mathcal{K}};\ \gamma(\mathbf{r})=\mathbf{r},\ \forall\gamma\in\Gamma\}$. For example, $\mathbf{0}\in\operatorname{Fix}(\Gamma_{\mathcal{X}})$, because $\gamma(\mathbf{0})=\mathbf{0}$ for any linear transformation $\gamma:\mathbb{R}^{\mathcal{K}}\longrightarrow\mathbb{R}^{\mathcal{K}}$.

Proposition 3.1. *Let* $\mathcal{X}\subset\{\pm1\}^{\mathcal{K}}$ *and suppose* $\operatorname{span}(\mathcal{X})=\mathbb{R}^{\mathcal{K}}$.

(a) *If* $\operatorname{Fix}(\Gamma_{\mathcal{X}})=\{\mathbf{0}\}$, *then* $\mathcal{X}$ *is McGarvey.*
(b) *In particular, if* $-\mathcal{X}=\mathcal{X}$, *then* $\mathcal{X}$ *is McGarvey.*

Clearly, $\mathcal{X}$ cannot be McGarvey unless $\operatorname{span}(\mathcal{X})=\mathbb{R}^{\mathcal{K}}$. One advantage of Proposition 3.1 over Theorem 1.3(b2) is that it is generally easier to verify that $\operatorname{span}(\mathcal{X})=\mathbb{R}^{\mathcal{K}}$ than it is to verify that $\mathbf{0}\in\operatorname{int}[\operatorname{conv}(\mathcal{X})]$. For instance, the next result is often sufficient.

Lemma 3.2. *Let* $\mathcal{X}\subseteq\{\pm1\}^{\mathcal{K}}$. *Suppose that, for every* $j\in\mathcal{K}$, *there exist* $\mathbf{x},\mathbf{y}\in\mathcal{X}$ *such that* $x_j\neq y_j$, *but* $x_k=y_k$ *for all* $k\in\mathcal{K}\setminus\{j\}$. *Then* $\operatorname{int}[\operatorname{conv}(\mathcal{X})]\neq\emptyset$, *and thus* $\operatorname{span}(\mathcal{X})=\mathbb{R}^{\mathcal{K}}$.

Example 3.3 (*Preference Aggregation*)**.** As discussed in the introduction, let $\mathcal{A}$ be a set with $|\mathcal{A}|\geq3$, and let $\mathcal{K}\subset\mathcal{A}\times\mathcal{A}$ be a subset containing exactly one of (a,b) or (b,a) for each $a\neq b\in\mathcal{A}$, so that $\{\pm1\}^{\mathcal{K}}$ represents the set of all tournaments on $\mathcal{A}$. Let $\mathcal{X}^{\rm pr}_{\mathcal{A}}\subset\{\pm1\}^{\mathcal{K}}$ be the space of preference orders on $\mathcal{A}$. For any $(a,b)\in\mathcal{K}$, there exist $\mathbf{x},\mathbf{y}\in\mathcal{X}^{\rm pr}_{\mathcal{N}}$ such that $x_{a,b}\neq y_{a,b}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. (For example, let $\mathbf{x}$ represent an ordering of the form $a\prec b\prec c_3\prec c_4\prec\cdots\prec c_N$, and let $\mathbf{y}$ represent the ordering $b\prec a\prec c_3\prec c_4\prec\cdots\prec c_N$.) Thus, Lemma 3.2 implies that $\operatorname{span}(\mathcal{X}^{\rm pr}_{\mathcal{A}})=\mathbb{R}^{\mathcal{K}}$.

Clearly, $-\mathcal{X}^{\rm pr}_{\mathcal{A}}=\mathcal{X}^{\rm pr}_{\mathcal{A}}$ (if $\mathbf{x}$ represents the ordering $a_1\prec a_2\prec\cdots\prec a_N$, then $-\mathbf{x}$ represents the ordering $a_1\succ a_2\succ\cdots\succ a_N$). Thus, Proposition 3.1(b) implies McGarvey's original result: $\mathcal{X}^{\rm pr}_{\mathcal{A}}$ is McGarvey. □

Example 3.4 (*Additively Separable Preferences*)**.** Let $\mathcal{A}_1,\ldots,\mathcal{A}_L$ be finite sets, and let $\mathcal{A}:=\mathcal{A}_1\times\cdots\times\mathcal{A}_L$. A binary relation "$\succ$" on $\mathcal{A}$ is *separable* if, for any $\mathcal{M}\subset[1\ldots L]$ and every $\mathbf{a},\mathbf{b},\mathbf{a}',\mathbf{b}'\in\mathcal{A}$, if

$$\begin{aligned}&\mathbf{a}_{\mathcal{M}}=\mathbf{a}'_{\mathcal{M}},\qquad \mathbf{a}_{\mathcal{M}^{\complement}}=\mathbf{b}_{\mathcal{M}^{\complement}},\\&\mathbf{b}_{\mathcal{M}}=\mathbf{b}'_{\mathcal{M}},\quad\text{and}\quad \mathbf{a}'_{\mathcal{M}^{\complement}}=\mathbf{b}'_{\mathcal{M}^{\complement}},\end{aligned}$$

then $(\mathbf{a}\succ\mathbf{b})\iff(\mathbf{a}'\succ\mathbf{b}')$. To put it another way, for any $\mathcal{M}\subseteq[1\ldots L]$, define $\mathcal{A}_{\mathcal{M}}:=\prod_{m\in\mathcal{M}}\mathcal{A}_m$ and $\mathcal{A}_{\mathcal{M}^{\complement}}:=\prod_{n\notin\mathcal{M}}\mathcal{A}_n$. Then "$\succ$" is separable if and only if, for each $\mathcal{M}\subseteq[1\ldots L]$, there is a binary relation "$\underset{\mathcal{M}}{\succ}$" on $\mathcal{A}_{\mathcal{M}}$ such that, for all $\mathbf{a}_{\mathcal{M}},\mathbf{b}_{\mathcal{M}}\in\mathcal{A}_{\mathcal{M}}$ and all $\mathbf{c}_{\mathcal{M}^{\complement}}\in\mathcal{A}_{\mathcal{M}^{\complement}}$, we have $((\mathbf{a}_{\mathcal{M}},\mathbf{c}_{\mathcal{M}^{\complement}})\succ(\mathbf{b}_{\mathcal{M}},\mathbf{c}_{\mathcal{M}^{\complement}}))\iff(\mathbf{a}_{\mathcal{M}}\underset{\mathcal{M}}{\succ}\mathbf{b}_{\mathcal{M}})$. Separability is an important domain restriction in certain voting models, such as the study of logrolling [8,24].

Define $\mathcal{K}\subset\mathcal{A}\times\mathcal{A}$ as in Example 3.3, so that $\{\pm1\}^{\mathcal{K}}$ represents the space of all tournaments on $\mathcal{A}$. Let $\mathcal{X}^{\rm sep}_{\mathcal{A}}\subset\{\pm1\}^{\mathcal{K}}$ be the set of all separable tournaments over $\mathcal{A}$; thus, $\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}}:=\mathcal{X}^{\rm sep}_{\mathcal{A}}\cap\mathcal{X}^{\rm pr}_{\mathcal{A}}$ is the set of all separable preference orders. It is easy to check that $\operatorname{maj}(\mathcal{X}^{\rm sep}_{\mathcal{A}})=\mathcal{X}^{\rm sep}_{\mathcal{A}}$; thus $\operatorname{med}^{\infty}(\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}})\subseteq\mathcal{X}^{\rm sep}_{\mathcal{A}}$. In fact, Hollard and le Breton [8] and Vidu [24] have shown that $\operatorname{maj}(\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}})=\mathcal{X}^{\rm sep}_{\mathcal{A}}$ (thus, $\operatorname{maj}(\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}})=\operatorname{med}^{\infty}(\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}})=\mathcal{X}^{\rm sep}_{\mathcal{A}}$). Thus, $\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}}$ is actually *not* McGarvey, according to our definition.

The problem here is that $\mathcal{X}^{\rm sep}_{\mathcal{A}}$ has many 'redundant' coordinates, which must always agree in value because of the separability constraints. Suppose we eliminate redundant coordinates, to obtain a subset $\widetilde{\mathcal{K}}\subset\mathcal{K}$ such that no two coordinates of $\widetilde{\mathcal{K}}$ are related by separability constraints. If $\pi_{\widetilde{\mathcal{K}}}:\{\pm1\}^{\mathcal{K}}\longrightarrow\{\pm1\}^{\widetilde{\mathcal{K}}}$ is the coordinate projection, then its restriction $\pi_{\widetilde{\mathcal{K}}}:\mathcal{X}^{\rm sep}_{\mathcal{A}}\longrightarrow\{\pm1\}^{\widetilde{\mathcal{K}}}$ is a bijection. Thus, if $\widetilde{\mathcal{X}}^{\rm sep_{pr}}_{\mathcal{A}}:=\pi_{\widetilde{\mathcal{K}}}(\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}})$, then the result of Hollard and le Breton [8] and Vidu [24] is equivalent to the assertion that $\operatorname{maj}(\widetilde{\mathcal{X}}^{\rm sep_{pr}}_{\mathcal{A}})=\{\pm1\}^{\widetilde{\mathcal{K}}}$.

In the Appendix, we give one possible definition of $\widetilde{\mathcal{K}}$, and then use Proposition 3.1(b) to show that $\operatorname{maj}(\widetilde{\mathcal{X}}^{\rm sep_{pr}}_{\mathcal{A}})=\{\pm1\}^{\widetilde{\mathcal{K}}}$. In fact, we prove a stronger assertion. A preference order "$\succ$" is *additively separable* if there is a set of 'utility functions' $u_\ell:\mathcal{A}_\ell\longrightarrow\mathbb{R}$ for $\ell\in[1\ldots L]$ such that, for all $\mathbf{a},\mathbf{b}\in\mathcal{A}$, we have $(\mathbf{a}\succ\mathbf{b})\iff(\sum_{\ell=1}^{L}u_\ell(a_\ell)>\sum_{\ell=1}^{L}u_\ell(b_\ell))$. Let $\mathcal{X}^{\rm add}_{\mathcal{A}}\subseteq\mathcal{X}^{\rm sep_{pr}}_{\mathcal{A}}$ be the space of additively separable preferences, and let $\widetilde{\mathcal{X}}^{\rm add}_{\mathcal{A}}:=\pi_{\widetilde{\mathcal{K}}}(\mathcal{X}^{\rm add}_{\mathcal{A}})$. Observe that $-\mathcal{X}^{\rm add}_{\mathcal{A}}=\mathcal{X}^{\rm add}_{\mathcal{A}}$

(if $\mathbf{x} \in \mathcal{X}^{\text{add}}_{\mathcal{A}}$ represents the preference order "$\succ$" defined by utility functions $u_1, \ldots, u_L$, then $-\mathbf{x} \in \mathcal{X}^{\text{add}}_{\mathcal{A}}$ represents the reversed preference order "$\prec$", which is defined by utility functions $-u_1, \ldots, -u_L$). It follows that $-\widetilde{\mathcal{X}}^{\text{add}}_{\mathcal{A}} = \widetilde{\mathcal{X}}^{\text{add}}_{\mathcal{A}}$. In the Appendix, we use Lemma 3.2 to show that $\text{span}(\widetilde{\mathcal{X}}^{\text{add}}_{\mathcal{A}}) = \mathbb{R}^{\widetilde{\mathcal{K}}}$. Thus, Proposition 3.1(b) says $\text{maj}(\widetilde{\mathcal{X}}^{\text{add}}_{\mathcal{A}}) = \{\pm 1\}^{\widetilde{\mathcal{K}}}$. It follows that $\text{maj}(\widetilde{\mathcal{X}}^{\text{sepr}}_{\mathcal{A}}) = \{\pm 1\}^{\widetilde{\mathcal{K}}}$ (and $\text{maj}(\mathcal{X}^{\text{add}}_{\mathcal{A}}) = \mathcal{X}^{\text{sep}}_{\mathcal{A}}$). $\square$

Example 3.5 (*Linear Classification*). Let $D \in \mathbb{N}$, and let $\mathcal{K} \subset \mathbb{R}^D$ be a finite set of points. For any nonzero $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ and $q \in \mathbb{R}$, let $\mathcal{H}^{\mathbf{r}}_q := \{\mathbf{k} \in \mathcal{K};\ \mathbf{r} \bullet \mathbf{k} \leq q\}$ (the intersection of $\mathcal{K}$ with a half-space in $\mathbb{R}^D$). Then define $\mathbf{x}^{\mathbf{r}}_q \in \{\pm 1\}^{\mathcal{K}}$ by $(x^{\mathbf{r}}_q)_{\mathbf{k}} = 1$ if $\mathbf{k} \in \mathcal{H}^{\mathbf{r}}_q$, and $(x^{\mathbf{r}}_q)_{\mathbf{k}} = -1$ if $\mathbf{k} \notin \mathcal{H}^{\mathbf{r}}_q$. Let $\mathcal{X} := \{\mathbf{x}^{\mathbf{r}}_q;\ \mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ and $q \in \mathbb{R}\}$. Intuitively, each element of $\mathcal{X}$ represents a 'classification' of the elements of $\mathcal{K}$ into two subsets separated by an affine hyperplane in $\mathbb{R}^D$.

Note that $-\mathcal{X} = \mathcal{X}$. To see this, let $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ and $q \in \mathcal{K}$. We have $-\mathbf{x}^{\mathbf{r}}_q = \mathbf{x}^{-\mathbf{r}}_{-q}$ if there is no $\mathbf{k} \in \mathcal{K}$ with $\mathbf{r} \bullet \mathbf{k} = q$. If there *is* such a $\mathbf{k}$, then we have $-\mathbf{x}^{\mathbf{r}}_q = \mathbf{x}^{-\mathbf{r}}_{-q'}$ for any $q' > q$ sufficiently close to q (because $\mathcal{K}$ is finite).

In the Appendix, we prove $\text{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$. Thus, Proposition 3.1(b) implies that $\mathcal{X}$ is McGarvey. $\square$

3.1. Symmetric sets of tournaments

Let $\mathcal{A}$ and $\mathcal{K}$ be as in Example 3.3. Let $\Pi_{\mathcal{A}}$ be the group of all permutations of $\mathcal{A}$; then $\Pi_{\mathcal{A}}$ acts on the set of tournaments on $\mathcal{A}$ by permuting vertices in the obvious way. (Note: permutations of $\mathcal{A}$ do *not* correspond to permutations of $\mathcal{K}$.) If $\mathcal{T}$ is a collection of tournaments on $\mathcal{A}$, then we say $\mathcal{T}$ is *symmetric* if $\pi(\mathcal{T}) = \mathcal{T}$ for all $\pi \in \Pi_{\mathcal{A}}$. For any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, let $\mathbf{T}_{\mathbf{x}}$ be the tournament defined by $\mathbf{x}$. Define $\mathcal{X}_{\mathcal{T}} := \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}};\ \mathbf{T}_{\mathbf{x}} \in \mathcal{T}\}$. (For example, $\mathcal{X}^{\text{pr}}_{\mathcal{A}} = \mathcal{X}_{\mathcal{T}^{\text{pr}}}$ where $\mathcal{T}^{\text{pr}}$ is the set of all preference orders on $\mathcal{A}$. Observe that $\mathcal{T}^{\text{pr}}$ is symmetric.)

Let $\mathbf{T} \in \mathcal{T}$. Regard $\mathbf{T}$ as a digraph. For any $a \in \mathcal{A}$, let $\#\text{In}_a(\mathbf{T})$ be the number of edges going into vertex a, while $\#\text{Out}_a(\mathbf{T})$ is the number of edges coming out of a. (Thus, $\#\text{In}_a(\mathbf{T}) + \#\text{Out}_a(\mathbf{T}) = |\mathcal{A}| - 1$.) A *directed Eulerian circuit* on $\mathbf{T}$ is a directed path through $\mathbf{T}$ which crosses every directed edge (in the correct direction) exactly once, and which begins and ends at the same vertex. It is well known that $\mathbf{T}$ admits a directed Eulerian circuit if and only if $\#\text{In}_a(\mathbf{T}) = \#\text{Out}_a(\mathbf{T})$ for every $a \in \mathcal{A}$. Shelah [21] has recently proved the following generalization of McGarvey's theorem.

Proposition 3.6 (*[21]*). *Suppose* $|\mathcal{A}| \geq 3$. *Let* $\mathcal{T}$ *be a symmetric set of tournaments on* $\mathcal{A}$. *Then*

$$(\mathcal{X}_{\mathcal{T}} \text{ is McGarvey}) \iff (\text{There exists some } \mathbf{T} \in \mathcal{T} \text{ which does not admit a directed Eulerian circuit}).$$

In the Appendix, we give a simple proof of Proposition 3.6 as a consequence of Proposition 3.1(a). (Most of the work is devoted to showing that the right-hand side implies that $\text{span}(\mathcal{X}_{\mathcal{T}}) = \mathbb{R}^{\mathcal{K}}$.)

3.2. Coordinate permutations

Let $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^{\mathcal{K}}$, and let $\mathbb{R}\mathbf{1} \subset \mathbb{R}^{\mathcal{K}}$ be the linear subspace it generates.

Proposition 3.7. *Let* $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *and suppose* $\text{Fix}(\Gamma_{\mathcal{X}}) \subseteq \mathbb{R}\mathbf{1}$. *Then* $\mathcal{X}$ *is McGarvey if and only if* $\text{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$ *and there exist* $r < 0 < t \in \mathbb{R}$ *such that* $r\mathbf{1}, t\mathbf{1} \in \text{conv}(\mathcal{X})$.

A *coordinate permutation* of $\mathbb{R}^{\mathcal{K}}$ is a linear map $\gamma : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R}^{\mathcal{K}}$ which maps any vector $(r_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\mathcal{K}}$ to the vector $(r_{\pi(k)})_{k \in \mathcal{K}}$, for some fixed permutation $\pi : \mathcal{K} \longrightarrow \mathcal{K}$. The set of all coordinate permutations in $\Gamma_{\mathcal{X}}$ forms a subgroup, which is isomorphic to a group $\Pi_{\mathcal{X}}$ of permutations on $\mathcal{K}$ in the obvious fashion. We say that $\Pi_{\mathcal{X}}$ is *transitive* if, for any $j, k \in \mathcal{K}$, there is some $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(j) = k$. For any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, let $\#(\mathbf{x}) := \#\{k \in \mathcal{K};\ x_k = 1\}$.

Corollary 3.8. *Let* $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ *and suppose* $\Pi_{\mathcal{X}}$ *is transitive. Then* $\mathcal{X}$ *is McGarvey if and only if* $\text{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$ *and there exist* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ *with* $\#(\mathbf{x}) < K/2 < \#(\mathbf{y})$.

Example 3.9 (*Symmetric Binary Relations*). Let $\mathcal{N}$ be a set, and let $\mathcal{K}$ be the set of all subsets $\{n, m\} \subseteq \mathcal{N}$ containing exactly two elements. Interpret each element of $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ as encoding a symmetric, reflexive binary relation "$\sim$" (i.e. for any $\{n, m\} \in \mathcal{K}$, we have $n \sim m$ if $x_{n,m} = 1$ and $n \not\sim m$ if $x_{n,m} = -1$). For any permutation $\pi : \mathcal{N} \longrightarrow \mathcal{N}$, define $\pi_* : \mathcal{K} \longrightarrow \mathcal{K}$ by $\pi\{n, m\} := \{\pi(n), \pi(m)\}$ for all $\{n, m\} \in \mathcal{K}$. Let Π_* be the set of all such permutations; then Π_* acts transitively on $\mathcal{K}$ (for any $\{n_1, m_1\} \in \mathcal{K}$ and $\{n_2, m_2\} \in \mathcal{K}$, let $\pi : \mathcal{N} \longrightarrow \mathcal{N}$ be any permutation such that $\pi(n_1) = n_2$ and $\pi(m_1) = m_2$; then $\pi_*\{n_1, m_1\} = \{n_2, m_2\}$).

(a) (*Equivalence relations*). Let $\mathcal{X}^{\text{eq}}_{\mathcal{N}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of equivalence relations. Then $\Pi_{\mathcal{X}^{\text{eq}}_{\mathcal{N}}}$ is transitive because it contains Π_*.

For any $\{n, m\} \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}$ such that $x_{n,m} \neq y_{n,m}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. (For example: let $\mathbf{x}$ represent an equivalence relation where n and m are both in singleton equivalence classes, and let $\mathbf{y}$ represent the relation obtained from $\mathbf{x}$ by joining n and m together into one doubleton equivalence class). Thus, Lemma 3.2 implies that $\text{span}(\mathcal{X}^{\text{eq}}_{\mathcal{N}}) = \mathbb{R}^{\mathcal{K}}$.

Note that $\pm\mathbf{1} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}$ ($\mathbf{1}$ represents the 'complete' relation "$\sim$" such that $n \sim m$ for all $n, m \in \mathcal{N}$, whereas $-\mathbf{1}$ represents the 'trivial' relation such that $n \not\sim m$ for any $n \neq m \in \mathcal{N}$). Thus, Corollary 3.8 implies that $\mathcal{X}^{\text{eq}}_{\mathcal{N}}$ is McGarvey.

This result (and Example 3.3) do not really require Corollary 3.8; in fact, we can obtain more refined results about $\mathcal{X}^{\rm pr}_{\mathcal{A}}$ and $\mathcal{X}^{\rm eq}_{\mathcal{N}}$ by using special structural properties of these spaces which have nothing to do with symmetry *per se* (see Example 7.4 below). However, the next four examples do make essential use of symmetry.

(b) (*Restricted Equivalence Relations*). For any $\mathbf{x} \in \mathcal{X}^{\rm eq}_{\mathcal{N}}$, let $\mathrm{rank}(\mathbf{x})$ be the number of distinct equivalence classes of the relation defined by $\mathbf{x}$. Suppose $2 \le r < R \le N$, and let $\mathcal{X}^{\rm eq}_{\mathcal{N}}(r, R)$ be the set of all $\mathbf{x} \in \mathcal{X}^{\rm eq}_{\mathcal{N}}$ with $r \le \mathrm{rank}(\mathbf{x}) \le R$; this is the set of all equivalence relations on $\mathcal{N}$ satisfying certain constraints on the 'coarseness' or 'fineness' of the equivalence partition. Clearly $\Pi_{\mathcal{X}^{\rm eq}_{\mathcal{N}}(r,R)} \supseteq \Pi_*$, so it is transitive. One can show $\mathrm{span}[\mathcal{X}^{\rm eq}_{\mathcal{N}}(r, R)] = \mathbb{R}^{\mathcal{K}}$ through a very similar argument to example (a). Thus, we can apply Corollary 3.8. Define

$$\overline{r}(N) := N + 1 - \frac{1 + \sqrt{2N^2 - 2N + 1}}{2}.$$

(Thus, if N is large, then $\overline{r}(N) \approx N - N/\sqrt{2}$.) In the Appendix, we show that $\mathcal{X}^{\rm eq}_{\mathcal{N}}(r, R)$ is McGarvey if and only if $r < \overline{r}(N)$.

(c) (*Connected graphs*). We can also interpret any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ as encoding a *graph*. Let $\mathcal{X}^{\rm cnct}_{\mathcal{N}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of all elements of $\{\pm 1\}^{\mathcal{K}}$ representing connected graphs on $\mathcal{N}$. Then $\Pi_{\mathcal{X}^{\rm cnct}_{\mathcal{N}}}$ is transitive because it contains Π_*.

For any $\{n, m\} \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\rm cnct}_{\mathcal{N}}$ such that $x_{n,m} \neq y_{n,m}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. (For example: let $\mathbf{x}$ represent a connected graph where vertices n and m are *not* linked. Let $\mathbf{y}$ represent the graph obtained from $\mathbf{x}$ by adding a link from n to m). Thus, Lemma 3.2 implies that $\mathrm{span}(\mathcal{X}^{\rm cnct}_{\mathcal{N}}) = \mathbb{R}^{\mathcal{K}}$.

There exists $\mathbf{x} \in \mathcal{X}^{\rm cnct}_{\mathcal{N}}$ with $\#(\mathbf{x}) < K/2$ (for example, let $\mathbf{x}$ represent a graph where the elements of $\mathcal{N}$ are arranged in a loop —then $\#(\mathbf{x}) = |\mathcal{N}| < K/2$). There also exists $\mathbf{y} \in \mathcal{X}^{\rm cnct}_{\mathcal{N}}$ with $\#(\mathbf{y}) > K/2$ (for example: $\mathbf{1} \in \mathcal{X}^{\rm cnct}_{\mathcal{N}}$). Thus, Corollary 3.8 says that $\mathcal{X}^{\rm cnct}_{\mathcal{N}}$ is McGarvey.

(d) (*Trees*). A graph is a *tree* if it is connected but contains no loops. Let $\mathcal{X}^{\rm tree}_{\mathcal{N}} \subset \mathcal{X}^{\rm cnct}_{\mathcal{N}}$ be the space of all trees. Let $N := |\mathcal{N}|$; then $\#(\mathbf{x}) = N - 1$ for every $\mathbf{x} \in \mathcal{X}^{\rm tree}_{\mathcal{N}}$ (because every tree has exactly $N - 1$ activated edges). Thus, Corollary 3.8 implies that $\mathcal{X}^{\rm tree}_{\mathcal{N}}$ is *not* McGarvey.

Interestingly, however, $\mathcal{X}^{\rm tree}_{\mathcal{N}}$ *is* median-saturating. To see this, note that any loop in a graph must involve at least three activated edges, and if $|\mathcal{N}| \ge 4$, then any disconnected graph must have at least three deactivated edges. Thus, $\mathcal{W}_2(\mathcal{X}^{\rm tree}_{\mathcal{N}}) = \emptyset$; hence Proposition 1.1(b) implies that $\mathrm{med}^\infty(\mathcal{X}^{\rm tree}_{\mathcal{N}}) = \{\pm 1\}^{\mathcal{K}}$. Thus, Eq. (4) is false for $\mathcal{X}^{\rm tree}_{\mathcal{N}}$. □

An interesting open question: what is the correct analogue of Proposition 3.6 when $\mathcal{T}$ is a symmetric set of symmetric binary relations (i.e. graphs) on $\mathcal{A}$?

4. Comprehensive property spaces

For any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{K}}$, write $\mathbf{r} \le \mathbf{s}$ if $r_k \le s_k$ for all $k \in \mathcal{K}$. Write $\mathbf{r} \ll \mathbf{s}$ if $r_k < s_k$ for all $k \in \mathcal{K}$. The space $\mathcal{X}$ is *comprehensive* if, for all $\mathbf{x} \in \mathcal{X}$ and all $\mathbf{y} \in \{\pm 1\}^{\mathcal{K}}$, if $\mathbf{x} \le \mathbf{y}$, then $\mathbf{y} \in \mathcal{X}$ also.

Example 4.1. Let $\mathcal{K}$ be a set of 'candidates'. Each $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ represents a 'committee' drawn from $\mathcal{K}$. Suppose $\mathcal{X}$ is the set of all committees satisfying a certain minimum level of representation from certain subgroups of candidates (e.g. "at least 3 female committee members"), with no upper bounds on the size of the whole committee. Then $\mathcal{X}$ is comprehensive. □

Proposition 4.2. *Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be comprehensive. The following are equivalent:* (a) *$\mathcal{X}$ is McGarvey;* (b) *There exists $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$ with $\mathbf{c} \ll \mathbf{0}$;* (c) $-\mathbf{1} \in \mathrm{maj}(\mathcal{X})$.

Example 4.3. Suppose $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ is comprehensive and there is a subset $\mathcal{Y} \subseteq \mathcal{X}$ such that, for each $k \in \mathcal{K}$, we have $\#\{\mathbf{y} \in \mathcal{Y}; y_k = 1\} < |\mathcal{Y}|/2$. Let $\mathbf{c} := \frac{1}{|\mathcal{Y}|} \sum_{\mathbf{y} \in \mathcal{Y}} \mathbf{y}$; then $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$ and $\mathbf{c} \ll \mathbf{0}$; hence $\mathcal{X}$ is McGarvey. □

In comprehensive spaces, median saturation is substantially weaker than the McGarvey property.

Proposition 4.4. *Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be comprehensive. Then $\mathcal{X}$ is median-saturating if and only if, for every $j, k \in \mathcal{K}$, there exists $\mathbf{x} \in \mathcal{X}$ with $x_j = 0 = x_k$.*

Example 4.5. Let $K/2 \le M \le K - 2$, and let $\mathcal{X}^{\rm com}_{\ge M} := \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}; \#(\mathbf{x}) \ge M\}$. Then $\mathcal{X}^{\rm com}_{\ge M}$ is median-saturating (by Proposition 4.4) but not McGarvey (by Corollary 3.8); thus, Eq. (4) is false for $\mathcal{X}^{\rm com}_{\ge M}$. □

5. Truth-functional aggregation

Let $\mathcal{J}$ be a set of logically independent propositions, and let $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be some function. Let $\mathcal{K} := \mathcal{J} \sqcup \{0\}$, and define $\mathcal{X}_f := \{(\mathbf{x}, y); \mathbf{x} \in \{\pm 1\}^{\mathcal{J}} \text{ and } y = f(\mathbf{x})\}$; a subset of $\{\pm 1\}^{\mathcal{K}}$; this is called a *truth-functional space*; see [19,5].

Many truth-functional spaces are not McGarvey. For example, let $\& : \{\pm 1\}^2 \longrightarrow \{\pm 1\}$ be the Boolean 'and' operation (i.e. $\&(x_1, x_2) = 1$ if and only if $x_1 = 1 = x_2$; otherwise $\&(x_1, x_2) = -1$), and let $\mathcal{X}_\& \subset \{\pm 1\}^3$ be the corresponding truth-functional space. Then $\mathcal{X}_\&$ is not McGarvey. Indeed, $\mathcal{X}_\&$ is not even median-saturating (this follows from Proposition 1.1(b), because $\mathcal{W}_2(\mathcal{X}_\&)$ contains the forbidden word $(*, 0; 1)$).

Proposition 5.1. *Suppose $|\mathcal{J}| \geq 2$, and suppose $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ depends nontrivially on more than one $\mathcal{J}$-coordinate. If $\sum_{\mathbf{x}\in\{\pm 1\}^{\mathcal{J}}} f(\mathbf{x}) = 0$, then $\mathcal{X}_f$ is McGarvey.*

For example, let $\oplus : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be the J-ary 'exclusive or' function. That is: $\oplus(\mathbf{x}) = 1$ if and only if $\#\{j \in \mathcal{J}; x_j = 1\}$ is *odd*. Then $\mathcal{X}_{\oplus}$ is McGarvey.

Proposition 5.2. *Let $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be a truth function. Suppose $f^{-1}\{1\}$ and $f^{-1}\{-1\}$ are both McGarvey, when seen as subsets of $\{\pm 1\}^{\mathcal{J}}$. Then $\mathcal{X}_f$ is McGarvey.*

A truth function $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ is *monotone* if, for all $\mathbf{x}, \mathbf{y} \in \{\pm 1\}^{\mathcal{J}}$,

$$(f(\mathbf{x}) = 1 \text{ and } \mathbf{x} \leq \mathbf{y}) \Longrightarrow (f(\mathbf{y}) = 1).$$

Combining Propositions 4.2 and 5.2, we see that even monotone truth functions can be McGarvey.

Proposition 5.3. *Let $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ be monotone. Suppose that:*

1. *there exists $\mathcal{Y}_+ \subseteq f^{-1}\{1\}$ such that for each $j \in \mathcal{J}$, we have $\#\{\mathbf{y} \in \mathcal{Y}_+; y_j = 1\} < |\mathcal{Y}_+|/2$; and*
2. *there exists $\mathcal{Y}_- \subseteq f^{-1}\{-1\}$ such that for each $j \in \mathcal{J}$, we have $\#\{\mathbf{y} \in \mathcal{Y}_-; y_j = -1\} < |\mathcal{Y}_-|/2$.*

Then $\mathcal{X}_f$ is McGarvey.

For example, let $J \geq 7$ be odd, and let $I := (J-1)/2$. Let $\mathcal{J} := [1 \dots J]$. For any $n \in \mathbb{N}$, let $[n]$ be the unique element of $\mathcal{J}$ which is congruent to n, mod J. For all $j \in \mathcal{J}$, define $\mathbf{y}^j \in \{\pm 1\}^{\mathcal{J}}$ by $y^j_{[j+i]} = 1$ for all $i \in [1 \dots I]$, and $y^j_k = -1$ for all other $k \in \mathcal{J}$. Then define $f : \{\pm 1\}^{\mathcal{J}} \longrightarrow \{\pm 1\}$ as follows: $f(\mathbf{x}) = 1$ if and only if $\mathbf{x} \geq \mathbf{y}^j$ for some $j \in \mathcal{J}$. Then f is monotone, and the set $\mathcal{Y}_+ := \{\mathbf{y}^j; j \in \mathcal{J}\}$ satisfies hypothesis #1 of Proposition 5.3. On the other hand, let $\mathbf{z}^1 := (1, 1, -1, 1, 1, -1, 1, 1, -1, \dots)$, let $\mathbf{z}^2 := (1, -1, 1, 1, -1, 1, 1, -1, 1, \dots)$, and let $\mathbf{z}^3 := (-1, 1, 1, -1, 1, 1, -1, 1, 1, \dots)$. Then $\mathcal{Y}_- := \{\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3\}$ satisfies hypothesis #2 of Proposition 5.3. Thus, $\mathcal{X}_f$ is McGarvey.

6. Convexities

A *convexity structure* on $\mathcal{K}$ is a collection $\mathfrak{C}$ of subsets of $\mathcal{K}$ such that $\emptyset \in \mathfrak{C}$, $\mathcal{K} \in \mathfrak{C}$, and $\mathfrak{C}$ is closed under intersections [23]. Convexity structures often represent the 'convex' subsets of some geometry on $\mathcal{K}$.

Example 6.1. A *metric graph* is a graph where each edge is assigned a positive real number specifying its 'length'. Let $\mathcal{K}$ be the vertices of a metric graph. For any $j, k \in \mathcal{K}$, a *geodesic* between j and k is a minimal-length path from j to k. A subset $\mathcal{C} \subseteq \mathcal{K}$ is *convex* if it contains all the geodesics between any pair of points in $\mathcal{C}$. The set $\mathfrak{C}$ of all convex subsets of $\mathcal{K}$ is then a convexity structure on $\mathcal{K}$. □

For any $\mathcal{J} \subseteq \mathcal{K}$, define $\boldsymbol{\chi}^{\mathcal{J}} \in \{\pm 1\}^{\mathcal{K}}$ by $\chi^{\mathcal{J}}_j := 1$ for all $j \in \mathcal{J}$ and $\chi^{\mathcal{J}}_k := -1$ for all $k \in \mathcal{K} \setminus \mathcal{J}$. Given a convexity structure $\mathfrak{C}$ on $\mathcal{K}$, let $\mathcal{X}_{\mathfrak{C}} := \{\boldsymbol{\chi}^{\mathcal{C}}; \mathcal{C} \in \mathfrak{C}\}$. Thus, judgement aggregation on $\mathcal{X}_{\mathfrak{C}}$ is the problem of democratically selecting a convex subset of $\mathcal{K}$. (This problem arises, for example, when a jury wishes to award prizes to some selected subset of contestants according to some 'quality metric', or when an expert committee tries to classify an unfamiliar entity within a taxonomic hierarchy.)

Proposition 6.2. *Let $\mathfrak{C}$ be a convexity on $\mathcal{K}$, and let $\mathcal{X}_{\mathfrak{C}}$ be as above.*

(a) *For any $\mathcal{J} \subseteq \mathcal{K}$, $(\boldsymbol{\chi}^{\mathcal{J}} \in \operatorname{maj}(\mathcal{X}_{\mathfrak{C}})) \iff (\mathcal{J}$ is a union of elements of $\mathfrak{C})$.*
(b) *The following are equivalent:*
- **[i]** *$\mathcal{X}_{\mathfrak{C}}$ is McGarvey.*
- **[ii]** *$\mathcal{X}_{\mathfrak{C}}$ is median-saturating.*
- **[iii]** *$\mathfrak{C}$ includes all the singleton subsets of $\mathcal{K}$.*

For example, the metric graph convexity in Example 6.1 is McGarvey.

7. Stearns numbers

Even if $\mathcal{X}$ is McGarvey, the hypothesis of Theorem 1.3(b) leaves the possibility that we can only realize this McGarvey property using very precisely engineered profiles involving an astronomically large number of voters. This would greatly diminish the practical relevance of the McGarvey property. So we now ask: what is the smallest number of voters required to realize the McGarvey property of $\mathcal{X}$? This question was first studied by Stearns [22] for preference aggregation on $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$. For any $N \in \mathbb{N}$, let

$$\Delta^*_N(\mathcal{X}) := \left\{\mu \in \Delta^*(\mathcal{X}); \forall \mathbf{x} \in \mathcal{X}, \mu(\mathbf{x}) = \frac{n}{N} \text{ for some } n \in [0 \dots N]\right\}.$$

In other words, $\Delta^*_N(\mathcal{X})$ is the set of profiles which can be generated by a population of exactly N voters. Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be McGarvey. We define the *Stearns number* $S(\mathcal{X})$ to be the smallest integer such that, for any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, there exists some $N \leq S(\mathcal{X})$ and $\mu \in \Delta^*_N(\mathcal{X})$ with $\mathrm{maj}(\mu) = \mathbf{x}$. (Define $S(\mathcal{X}) := \infty$ if $\mathcal{X}$ is not McGarvey.) For example, if $A := |\mathcal{A}|$, then Stearns [22] showed that $0.55 \cdot A/\log(A) \leq S(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) \leq A + 2$. Erdös and Moser [6] refined Stearn's estimate by showing that $S(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) = \Theta(A/\log(A))$. We now investigate the Stearns numbers of other McGarvey spaces. For any $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$, let $\|\mathbf{r}\|_\infty := \sup_{k\in\mathcal{K}} |r_k|$. For any $\epsilon > 0$, let $\mathcal{B}(\epsilon) := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}}; \|\mathbf{r}\|_\infty \leq \epsilon\}$. For any $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$, let $\sigma(\mathcal{X}) := \min\{N \in \mathbb{N}; \mathcal{B}(\frac{1}{N}) \subseteq \mathrm{conv}(\mathcal{X})\}$. Note that $\sigma(\mathcal{X}) < \infty$ if and only if $\mathcal{X}$ is McGarvey. Thus, the next result can be seen as a 'quantitative' refinement of Theorem 1.3.

Theorem 7.1. *For any $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$, we have $\sigma(\mathcal{X}) \leq S(\mathcal{X}) \leq 4(K + 1)\sigma(\mathcal{X})$.*

The upper bound in Theorem 7.1 is an overestimate, in general. For example, Alon [1] has shown that $\sigma(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) = \Theta(\sqrt{A})$; and in the case of $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$, we have $K := A(A - 1)/2$; thus Theorem 7.1 yields $S(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}) \leq \mathcal{O}(A^{5/2})$, which is much worse than the estimate of $\Theta(A/\log(A))$ obtained by Erdös and Moser [6]. Nevertheless, it may not be possible to improve the estimate in Theorem 7.1, without making further assumptions about the structure of $\mathcal{X}$. The next result provides some bounds on the size of $\sigma(\mathcal{X})$ and $S(\mathcal{X})$. For any $\mathbf{x}_1, \ldots, \mathbf{x}_K \in \{\pm 1\}^{\mathcal{K}}$, let $\delta(\mathbf{x}_1, \ldots, \mathbf{x}_K) := \min\{\|\mathbf{c}\|_\infty; \mathbf{c} \in \mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_K)\}$. Let $\delta(\mathcal{X}) := \min\{\delta(\mathbf{x}_1, \ldots, \mathbf{x}_K); \mathbf{x}_1, \ldots, \mathbf{x}_K \in \mathcal{X}$ and $\mathbf{0} \notin \mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_K)\}$. Finally, let $\delta(K) := \delta(\{\pm 1\}^K)$.

Proposition 7.2. *Let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$.*

(a) *If $\mathcal{X}$ is McGarvey, then $\sigma(\mathcal{X}) \leq \lceil 1/\delta(\mathcal{X})\rceil$.*
(b) *For every McGarvey $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$, we have $S(\mathcal{X}) \leq 4(K + 1)\lceil 1/\delta(K)\rceil$.*
However, there exist McGarvey $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ with $S(\mathcal{X}) \geq 1/\delta(K)$.
(c) $\frac{K^{K/2}}{2^{2K+\mathcal{O}(K)}} \leq \frac{1}{\delta(K)} \leq \frac{K^{2+K/2}}{2^{K-1}}$.

The inequalities in Proposition 7.2(c) are derived from inequalities obtained by Alon and Vũ [2] for the inverses of $\{0, 1\}$-matrices; these inequalities have many implications for the geometry of sub-polytopes of $\{\pm 1\}^{\mathcal{K}}$ [26, Section 5.2]. Proposition 7.2(b) and (c) imply that the Stearns numbers of some McGarvey spaces can be extremely large. However, for the McGarvey spaces typically encountered in practice, the Stearns numbers are often much smaller, as shown by the next result and following examples.

Proposition 7.3. **(a)** *If $\mathbf{1} \in \mathcal{X}$, and $\boldsymbol{\chi}^k \in \mathcal{X}$ for all $k \in \mathcal{K}$, then $S(\mathcal{X}) \leq 2K - 3$.*
(b) *Suppose that $-\mathbf{1} \in \mathcal{X}$, and suppose that, for all $k \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ such that $x_k = 1 = y_k$, but $\mathbf{x}$ and $\mathbf{y}$ differ in every other coordinate. Then $S(\mathcal{X}) \leq 2K + 1$.*
(c) *Suppose $-\mathcal{X} = \mathcal{X}$ and suppose that, for all $k \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ such that $x_k \neq y_k$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate. Then $S(\mathcal{X}) \leq 2K$.*

Example 7.4. (a) (*Convexities*). Let $\mathfrak{C}$ be a convexity on $\mathcal{K}$. Then $\mathbf{1} \in \mathcal{X}_{\mathfrak{C}}$ (because $\mathcal{K} \in \mathfrak{C}$). If $\mathcal{X}_{\mathfrak{C}}$ is McGarvey, then Proposition 6.2(b) says $\boldsymbol{\chi}^k \in \mathcal{X}$ for all $k \in \mathcal{K}$; thus, Proposition 7.3(a) says $S(\mathcal{X}_{\mathfrak{C}}) \leq 2K - 3$.

(b) (*Equivalence Relations*). Let $\mathcal{N}$ be a set, and let $\mathcal{K}$ and $\mathcal{X}^{\mathrm{eq}}_{\mathcal{N}} \subset \{\pm 1\}^{\mathcal{K}}$ be as in Example 3.9(a). Observe that $\mathbf{1} \in \mathcal{X}^{\mathrm{eq}}_{\mathcal{N}}$ (it represents the 'complete equivalence' relation such that $n \sim m$ for all $n, m \in \mathcal{N}$). Also, for all $\{n, m\} \in \mathcal{N}$, $\boldsymbol{\chi}^{n,m} \in \mathcal{X}^{\mathrm{eq}}_{\mathcal{N}}$ (it represents the equivalence relation such that $n \sim m$, but no other pair of elements are equivalent). Thus, Proposition 7.3(a) implies that $\mathcal{X}^{\mathrm{eq}}_{\mathcal{N}}$ is McGarvey, and $S(\mathcal{X}^{\mathrm{eq}}_{\mathcal{N}}) \leq N(N - 1) - 3$.

(c) (*Preorders*). Let $\mathcal{K} := \{(n, m) \in \mathcal{N} \times \mathcal{N}; n \neq m\}$. Thus, an element of $\{\pm 1\}^{\mathcal{K}}$ can represent a reflexive binary relation "$\preceq$" on $\mathcal{N}$. A *preorder* is a reflexive, transitive binary relation on $\mathcal{N}$ (note that we do not assume preorders are complete). Let $\mathcal{X}^{\mathrm{preo}}_{\mathcal{N}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of all preorders on $\mathcal{N}$. Thus, $\mathbf{1} \in \mathcal{X}^{\mathrm{preo}}_{\mathcal{N}}$ (it represents the relation of total indifference). Also, for all $(n, m) \in \mathcal{N}$, $\boldsymbol{\chi}^{n,m} \in \mathcal{X}^{\mathrm{preo}}_{\mathcal{N}}$ (it represents the preorder such that $n \preceq m$, but no other pair of elements are comparable). Thus, Proposition 7.3(a) implies $\mathcal{X}^{\mathrm{preo}}_{\mathcal{N}}$ is McGarvey, and $S(\mathcal{X}^{\mathrm{preo}}_{\mathcal{N}}) \leq 2N(N - 1) - 3$.

(d) (*Complete preorders*). Now let $\mathcal{X}^* \subset \mathcal{X}^{\mathrm{preo}}_{\mathcal{N}}$ be the set of all *complete* preorders. Then $\mathcal{X}^*$ is *not* McGarvey. Indeed, Example 1.2 shows that $\mathcal{X}^*$ is not even median-saturating.

(e) Let $\mathcal{X}$ be the 'linear classification' space from Example 3.5. We have already seen that $-\mathcal{X} = \mathcal{X}$. The proof that $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$ (in the Appendix) defines a linear ordering on $\mathcal{K}$ and then constructs a subset $\{\mathbf{x}^k\}_{k\in\mathcal{K}} \subset \mathcal{X}$ such that, for all $j, k \in \mathcal{K}$, if j is the immediate predecessor to k, then $\mathbf{x}^j$ and $\mathbf{x}^k$ differ only in coordinate k. Thus, Proposition 7.3(c) implies that $S(\mathcal{X}) \leq 2K$. Note that, in applications, K (which is the number of elements to be classified) can be quite large. Hence $S(\mathcal{X})$ may be similarly large. □

8. Conclusion

In this paper, we have investigated when the aggregation of judgements by propositionwise majority votes results in a complete loss of structure at the group level. For this to occur, at the individual level, any pairwise combination of judgements on specific propositions must be admissible; this yields the property of *median saturation*. We showed that, for many (but not all) median-saturated spaces, McGarvey's original result about preference aggregation generalizes, and a complete loss of structure in fact occurs.

Median saturation is obviously restrictive, and in many contexts, there are built-in constraints on the judgements on pairs of propositions. For instance, if incomplete preferences (i.e. asymmetric and transitive binary relations) are aggregated, then asymmetry imposes such a pairwise constraint, which will be preserved by pairwise majority voting. On the other hand, in analogy to McGarvey's original result on linear orders, one would expect asymmetry to be the *only* restriction on the binary relation that is preserved by majoritarian voting. That is, one would expect Eq. (4) to be true: $\mathrm{maj}(\mathcal{X}) = \mathrm{med}^{\infty}(\mathcal{X})$.

Let us call Eq. (4) the *Generalized McGarvey Property*. The investigation of conditions under which this property obtains is an important task for future research, because it frequently seems natural and plausible. For example, Hollard and le Breton [8] and Vidu [24] have shown that $\mathrm{maj}(\mathcal{X}_{\mathcal{A}}^{\mathrm{sepr}}) = \mathrm{med}^{\infty}(\mathcal{X}_{\mathcal{A}}^{\mathrm{sepr}})$, where $\mathcal{X}_{\mathcal{A}}^{\mathrm{sepr}}$ is the set of all separable preference orders over a Cartesian product of finite sets.[4] Theorem 1.3(a) implies that, like the McGarvey Property, the Generalized McGarvey Property is a property of convex polytopes in $\{\pm 1\}^{\mathcal{K}}$ —namely the property that $\mathrm{conv}(\mathcal{X})$ intersect the open orthant $\mathcal{O}_{\mathbf{x}}$ for every $\mathbf{x} \in \mathrm{med}^{\infty}(\mathcal{X})$. The further analysis of this property and its applications to specific types of aggregation problems will yield interesting new challenges and rewards.

Acknowledgements

We were privileged to receive two exceptionally thorough reports from an excellent referee, who spotted several subtle mistakes, made many very perceptive comments, and made us aware of Ref. [21] and Proposition 3.6, as well Refs. [10,9] and Proposition 2.3((a) and (c)). This research was partly supported by the NSERC grant #262620-2008.

Appendix. Proofs

Proof of Proposition 1.1. Part (b) follows immediately from (a). Part (a) follows (after some decryption) from Lemma I.6.20(1) on p. 130 of [23]. We will give another proof of part (a), using 'critical words'. For any $\mathcal{Y} \subseteq \{\pm 1\}^{\mathcal{K}}$, let $\mathcal{W}(\mathcal{Y})$ be the set of all $\mathcal{Y}$-forbidden words. A word $\mathbf{w} \in \mathcal{W}(\mathcal{Y})$ is $\mathcal{Y}$-*critical* if no proper subword of $\mathbf{w}$ is in $\mathcal{W}(\mathcal{Y})$. Let $\mathcal{W}^*(\mathcal{Y})$ be the set of $\mathcal{Y}$-critical words. For any $\mathcal{X}, \mathcal{Y} \subseteq \{\pm 1\}^{\mathcal{K}}$, we have:

$$(\mathcal{W}^*(\mathcal{Y}) \subseteq \mathcal{W}^*(\mathcal{X})) \Longrightarrow (\mathcal{W}(\mathcal{Y}) \subseteq \mathcal{W}(\mathcal{X})) \iff (\mathcal{X} \subseteq \mathcal{Y}). \tag{5}$$

Proposition 4.1 of [17] states:

$$(\mathcal{Y} \text{ is a median space}) \iff (\text{All } \mathcal{Y}\text{-critical words have order 2}). \tag{6}$$

Let $\mathcal{Y} := \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}};\ \mathbf{w} \not\sqsubset \mathbf{x},\ \forall \mathbf{w} \in \mathcal{W}_2(\mathcal{X})\}$. We must show that $\mathrm{med}^{\infty}(\mathcal{X}) = \mathcal{Y}$. By construction, $\mathcal{W}^*(\mathcal{Y}) = \mathcal{W}_2(\mathcal{X})$. Thus, every $\mathcal{Y}$-critical word has order 2, so statement (6) says $\mathcal{Y}$ is a median space. Also, $\mathcal{W}(\mathcal{Y}) \subseteq \mathcal{W}(\mathcal{X})$, so (5) implies $\mathcal{X} \subseteq \mathcal{Y}$. But by definition, $\mathrm{med}^{\infty}(\mathcal{X})$ is the smallest median space containing $\mathcal{X}$. Thus, $\mathrm{med}^{\infty}(\mathcal{X}) \subseteq \mathcal{Y}$.

To see the reverse inclusion, note that $\mathrm{med}^{\infty}(\mathcal{X})$ is a median space; thus, statement (6) says every $\mathrm{med}^{\infty}(\mathcal{X})$-critical word has order 2. However, $\mathcal{X} \subseteq \mathrm{med}^{\infty}(\mathcal{X})$, so (5) implies $\mathcal{W}[\mathrm{med}^{\infty}(\mathcal{X})] \subseteq \mathcal{W}(\mathcal{X})$. Thus, $\mathcal{W}^*[\mathrm{med}^{\infty}(\mathcal{X})] \subseteq \mathcal{W}_2(\mathcal{X}) = \mathcal{W}^*(\mathcal{Y})$. Thus, (5) implies that $\mathcal{Y} \subseteq \mathrm{med}^{\infty}(\mathcal{X})$. Thus, $\mathcal{Y} = \mathrm{med}^{\infty}(\mathcal{X})$. □

Proof of Theorem 1.3. (a) Let $\mu \in \Delta^*(\mathcal{X})$. For all $k \in \mathcal{K}$, define $\widetilde{\mu}_k$ as in Eq. (1), and let $\widetilde{\mu} := (\widetilde{\mu}_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\mathcal{K}}$. Let $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ be the unique element such that $\widetilde{\mu} \in \mathcal{O}_{\mathbf{x}}$; then Eq. (2) implies that $\mathrm{maj}(\mu) = \mathbf{x}$.

If we treat $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ as a subset of $\mathbb{R}^{\mathcal{K}}$, then $\widetilde{\mu} := \sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x})\mathbf{x}$; thus, $\mu \in \mathrm{conv}(\mathcal{X})$. Furthermore, every element of $\mathrm{conv}(\mathcal{X})$ can be represented in this way. Thus, for any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$,

$$(\mathbf{x} \in \mathrm{maj}(\mathcal{X})) \iff (\exists \mu \in \Delta^*(\mathcal{X}) \text{ such that } \widetilde{\mu} \in \mathcal{O}_{\mathbf{x}}) \iff (\mathrm{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}} \neq \emptyset).$$

(b) "(b2) $\iff$ (b3)" The Separating Hyperplane Theorem says that $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{X})]$ if and only if, for all nonzero $\mathbf{z} \in \mathbb{R}^{\mathcal{K}}$, there exists $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$ such that $\mathbf{z} \bullet \mathbf{c} > 0$. This, in turn, occurs if and only if there exists $\mathbf{x} \in \mathcal{X}$ such that $\mathbf{z} \bullet \mathbf{x} > 0$ (because $\mathcal{X}$ is the set of extreme points of $\mathrm{conv}(\mathcal{X})$).

"(b2) $\iff$ (b5)" is immediate.

"(b1) $\Longleftarrow$ (b2)" If $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{X})]$, then $\mathrm{conv}(\mathcal{X})$ intersects every open orthant of $\mathbb{R}^{\mathcal{K}}$, so (a) implies that $\mathrm{maj}(\mathcal{X}) = \{\pm 1\}^{\mathcal{K}}$.

"(b1) $\Longrightarrow$ (b2)" (by contrapositive) $\mathrm{int}[\mathrm{conv}(\mathcal{X})]$ is an open convex subset of $\mathbb{R}^{\mathcal{K}}$. Suppose $\mathbf{0} \notin \mathrm{int}[\mathrm{conv}(\mathcal{X})]$. Then the Separating Hyperplane Theorem says there is some vector $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ such that $\mathbf{r} \bullet \mathbf{c} < 0$ for all $\mathbf{c} \in \mathrm{int}[\mathrm{conv}(\mathcal{X})]$. Pick $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ such that the open orthant $\mathcal{O}_{\mathbf{x}}$ contains $\mathbf{r}$ (if $\mathbf{r}$ sits on a boundary between two or more orthants, then pick one). Then we must have $\mathrm{int}[\mathrm{conv}(\mathcal{X})] \cap \mathcal{O}_{\mathbf{x}} = \emptyset$. Thus, $\mathrm{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}} = \emptyset$ (because $\mathrm{conv}(\mathcal{X})$ is the closure of $\mathrm{int}[\mathrm{conv}(\mathcal{X})]$, and $\mathcal{O}_{\mathbf{x}}$ is an open set). Thus, part (a) implies that $\mathbf{x} \notin \mathrm{maj}(\mathcal{X})$; hence $\mathcal{X}$ is not McGarvey.

"(b4) $\Longrightarrow$ (b2)" Suppose $\mathbf{0} = \widetilde{\mu}$ for some $\mu \in \Delta(\mathcal{X})$ such that $\mu(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$. Then for any $\mathbf{x} \in \mathcal{X}$, we have

$$-\mathbf{x} = \frac{1}{\mu(\mathbf{x})} \sum_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mu(\mathbf{y}) \cdot \mathbf{y}, \tag{7}$$

which is a strictly positive linear combination of the elements in $\mathcal{X} \setminus \{\mathbf{x}\}$.

[4] This example is quite special, however. By eliminating redundant coordinates, we saw in Example 3.4 that the results of [8,24] can be reformulated as the 'classical' McGarvey property in a lower-dimensional hypercube.

Fix nonzero $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$. We can write $\mathbf{r} = \sum_{\mathbf{x}\in\mathcal{X}} s_{\mathbf{x}}\mathbf{x}$ for some real-valued coefficients $\{s_{\mathbf{x}}\}_{\mathbf{x}\in\mathcal{X}}$ (because $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$). For any $\mathbf{x} \in \mathcal{X}$, if $s_{\mathbf{x}} < 0$, then replace the term "$s_{\mathbf{x}}\mathbf{x}$" with $-s_{\mathbf{x}}$ times the right side of Eq. (7). In this way, we can write $\mathbf{r} = \sum_{\mathbf{x}\in\mathcal{X}} s'_{\mathbf{x}}\mathbf{x}$ for some *positive* coefficients $\{s'_{\mathbf{x}}\}_{\mathbf{x}\in\mathcal{X}}$. Now let $S := \sum_{\mathbf{x}\in\mathcal{X}} s'_{\mathbf{x}}$. Then $0 < S < \infty$, and $\mathbf{r}/S \in \mathrm{conv}(\mathcal{X})$.

Thus, for any nonzero $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$, the ray from $\mathbf{0}$ through $\mathbf{r}$ passes through $\mathrm{conv}(\mathcal{X})$ at some point. Since $\mathrm{conv}(\mathcal{X})$ is convex, this implies that $\mathrm{conv}(\mathcal{X})$ contains a neighbourhood around $\mathbf{0}$.

"(b2) $\Longrightarrow$ (b4)" If $\mathrm{int}[\mathrm{conv}(\mathcal{X})] \neq \emptyset$, then $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$. Now, let $\nu \in \Delta^*(\mathcal{X})$ be any profile such that $\nu(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$. Since $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{X})]$, there exists some $\epsilon > 0$ such that $-\epsilon\widetilde{\nu} \in \mathrm{conv}(\mathcal{X})$, so find some $\eta \in \Delta^*(\mathcal{X})$ such that $\widetilde{\eta} = -\epsilon\widetilde{\nu}$. Now define $\mu := (\frac{\epsilon}{1+\epsilon})\nu + (\frac{1}{1+\epsilon})\eta$. Then $\mu \in \Delta(\mathcal{X})$, and $\widetilde{\mu} := (\frac{\epsilon}{1+\epsilon})\widetilde{\nu} + (\frac{1}{1+\epsilon})\widetilde{\eta} = \mathbf{0}$. Finally, $\mu(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$, because $\nu(\mathbf{x}) > 0$ and $\eta(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathcal{X}$. □

Proof of Proposition 2.1. (a) Let $M := \min\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is McGarvey$\}$.

"$M \geq K + 1$": Suppose $|\mathcal{X}| = J \leq K$. Let $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^J\}$. Define $\mathbf{y}^j := \mathbf{x}^j - \mathbf{x}^J$ for all $j \in [1 \ldots J-1]$, and let $\mathcal{Y}$ be the linear subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\{\mathbf{y}^1, \ldots, \mathbf{y}^{J-1}\}$. Then $\dim(\mathcal{Y}) \leq J - 1 < K$. However, $\mathrm{conv}(\mathcal{X}) \subset \mathcal{Y} + \mathbf{x}^J$; thus, $\mathrm{int}[\mathrm{conv}(\mathcal{X})] = \emptyset$, so $\mathcal{X}$ is not McGarvey.

"$M \leq K + 1$": Let $\mathbf{1} := (1, 1, \ldots, 1)$. For all $k \in \mathcal{K}$, define $\boldsymbol{\chi}^k \in \{\pm 1\}^{\mathcal{K}}$ as we did prior to Proposition 6.2. Let $\mathcal{X} := \{\boldsymbol{\chi}^k\}_{k\in\mathcal{K}} \sqcup \{\mathbf{1}\}$. Then $|\mathcal{X}| = K + 1$, and it is clear that $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$. We have

$$\left(\frac{K-2}{2K-2}\right)\mathbf{1} + \left(\frac{1}{2K-2}\right)\sum_{k\in\mathcal{K}} \boldsymbol{\chi}^k = \left(\frac{K-2}{2K-2}\right)\mathbf{1} - \left(\frac{K-2}{2K-2}\right)\mathbf{1} = \mathbf{0},$$

verifying condition (b4) of Theorem 1.3. Thus, $\mathcal{X}$ is McGarvey.

(b) Let $M := \max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is minimal McGarvey$\}$.

"$M \geq 2K$" follows from Example 2.2. To see "$M \leq 2K$", let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be McGarvey. Then Theorem 1.3(b2) says $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{X})]$.

Claim 1. *There exists some $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| \leq 2K$ such that $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{Y})]$.*

Proof. For any nonzero $\mathbf{v} \in \mathbb{R}^{\mathcal{K}}$, consider the line $\mathcal{L}_{\mathbf{v}} := \{r\mathbf{v};\ r \in \mathbb{R}\}$. This line intersects the boundary of $\mathrm{conv}(\mathcal{X})$ in exactly two places — say at $\mathbf{u} = -s\mathbf{v}$ and $\mathbf{w} = t\mathbf{v}$, for some $-s < 0 < t$. For a generic choice of $\mathbf{v} \in \mathbb{R}^{\mathcal{K}}$, the points $\mathbf{u}$ and $\mathbf{w}$ are each contained in the relative interior of some $(K-1)$-dimensional face of $\mathrm{conv}(\mathcal{X})$ — that is, there are sets $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_K\} \subseteq \mathcal{X}$ and $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_K\} \subseteq \mathcal{X}$, such that $\mathrm{conv}(\mathcal{U})$ and $\mathrm{conv}(\mathcal{W})$ each have dimension $(K-1)$, and such that $\mathbf{u} = \sum_{k=1}^{K} q_k\mathbf{u}_k$ and $\mathbf{w} = \sum_{k=1}^{K} r_k\mathbf{w}_k$, for some $q_1, \ldots, q_K, r_1, \ldots, r_K > 0$ with $\sum_{k=1}^{K} q_k = 1 = \sum_{k=1}^{K} r_k$. Let $\mathcal{Y} := \mathcal{U} \cup \mathcal{W}$. Then $\mathrm{conv}(\mathcal{Y})$ contains the $(K-1)$-dimensional sets $\mathrm{conv}(\mathcal{U})$ and $\mathrm{conv}(\mathcal{W})$, and it also contains two different points on the line $\mathcal{L}$ transversal to these sets (because $\mathrm{conv}(\mathcal{U})$ and $\mathrm{conv}(\mathcal{W})$ intersect $\mathcal{L}$ at two different points). Thus $\mathrm{conv}(\mathcal{Y})$ must have dimension K (hence, nonempty interior). Furthermore, $|\mathcal{Y}| \leq |\mathcal{U}| + |\mathcal{W}| = 2K$. Let $R := \frac{1}{s} + \frac{1}{t}$, let $S := \frac{1}{sR} > 0$ and let $T := \frac{1}{tR} > 0$. Then $S + T = 1$, and

$$\sum_{k=1}^{K} Sq_k\mathbf{u}_k + \sum_{k=1}^{K} Tr_k\mathbf{u}_k = S\sum_{k=1}^{K} q_k\mathbf{u}_k + T\sum_{k=1}^{K} r_k\mathbf{u}_k = \frac{-s\mathbf{v}}{sR} + \frac{t\mathbf{v}}{tR} = \frac{-\mathbf{v}}{R} + \frac{\mathbf{v}}{R} = \mathbf{0}.$$

By construction, we have $Sq_1, \ldots, Sq_K, Tr_1, \ldots, Tr_K > 0$, and $\sum_{k=1}^{K} Sq_k + \sum_{k=1}^{K} Tr_k = 1$. Thus, $\mathbf{0}$ is a strictly positive convex combination of the elements of $\mathcal{Y}$, so $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{Y})]$, as claimed. ◇ claim 1

If $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{Y})]$, then Theorem 1.3(b2) implies that $\mathcal{Y}$ is McGarvey. But if $\mathcal{X}$ is *minimal* McGarvey, then this means that $\mathcal{Y} = \mathcal{X}$. Thus, $|\mathcal{X}| \leq 2K$, as claimed. □

Remark. The proof of Claim 1 in Proposition 2.1(b) easily generalizes to prove the following 'relative interior' version of Carathéodory's theorem: *Let $\mathcal{X} \subset \mathbb{R}^K$ be finite, let $\dim(\mathrm{conv}(\mathcal{X})) = D \leq K$, and let $\mathbf{x}$ be in the relative interior of $\mathrm{conv}(\mathcal{X})$. Then there exists some $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| \leq 2D$ such that $\mathbf{x}$ is in the relative interior of $\mathrm{conv}(\mathcal{Y})$.*

Proof of Example 2.2. We must show that $\mathcal{X}$ is McGarvey, but no proper subset of $\mathcal{X}$ is McGarvey.

$\mathcal{X}$ is McGarvey. Clearly, $2\boldsymbol{\chi}^j \in (\mathcal{X} - \mathcal{X})$ for all $j \in \mathcal{K}$. Thus, $\mathrm{span}(\mathcal{X} - \mathcal{X}) = \mathbb{R}^{\mathcal{K}}$, so $\mathrm{int}[\mathrm{conv}(\mathcal{X})] \neq \emptyset$.

Recall from Section 3 that $\Pi_{\mathcal{X}}$ is the set of coordinate permutation symmetries of $\mathcal{X}$. In this case, $\Pi_{\mathcal{X}}$ contains every possible permutation of $\mathcal{K}$, so $\Pi_{\mathcal{X}}$ is transitive. Clearly $\#(\boldsymbol{\chi}^j) = 1 < K/2$, whereas $\#(-\boldsymbol{\chi}^j) = K - 1 > K/2$. Thus, Corollary 3.8 implies that $\mathcal{X}$ is McGarvey.

No proper subset of $\mathcal{X}$ is McGarvey. Suppose $\mathcal{K} := [1 \ldots K]$. Let $\mathcal{Y} := \mathcal{X} \setminus \{\boldsymbol{\chi}^1\}$. To see that $\mathcal{Y}$ is not McGarvey, let $\mathbf{z} := (K-3; -1, -1, \ldots, -1)$; then $\mathbf{z} \bullet \mathbf{y} \leq 0$ for all $\mathbf{y} \in \mathcal{Y}$, violating condition (b3) of Theorem 1.3(b). Thus, $\mathcal{Y}$ is not McGarvey.

A similar argument shows that $\mathcal{X} \setminus \{\boldsymbol{\chi}^k\}$ and $\mathcal{X} \setminus \{-\boldsymbol{\chi}^k\}$ are not McGarvey, for any $k \in \mathcal{K}$. □

Proof of Proposition 2.3. (a) Let $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^M\} \in \{\pm 1\}^K$. For all $k \in [1 \ldots K]$ and $s \in \{\pm 1\}$, let $\mathcal{A}_k^s := \{m \in [1 \ldots M];\ x_k^m = s\}$. Then $\mathcal{A}_1^{\pm 1}, \ldots, \mathcal{A}_K^{\pm 1} \subseteq [1 \ldots M]$, and for any $j, k \in [1 \ldots K]$ and $s_j, s_k \in \{\pm 1\}$, we have $\mathcal{A}_j^{s_j} \cap \mathcal{A}_k^{s_k} \neq \emptyset$ if and only if there exists some $\mathbf{x}^m \in \mathcal{X}$ such that $(x_j^m, x_k^m) = (s_j, s_k)$. Thus,

$$\begin{aligned}(\mathcal{X} \text{ is median-saturating}) &\underset{(*)}{\iff} (\mathcal{W}_2(\mathcal{X}) = \emptyset) \\ &\iff (\mathcal{A}_j^{s_j} \cap \mathcal{A}_k^{s_k} \neq \emptyset, \text{ for all } j, k \in [1 \ldots K] \text{ and } s_j, s_k \in \{\pm 1\}),\end{aligned} \tag{8}$$

where $(*)$ is by Proposition 1.1(b). The sets $\mathcal{A}_1^{+1}, \ldots, \mathcal{A}_K^{+1}$ are called *independent* if the condition on the right-hand side of statement (8) is satisfied.

Conversely, given any independent sets $\mathcal{A}_1^{+1}, \ldots, \mathcal{A}_K^{+1} \subseteq [1 \ldots M]$, if we define $\mathcal{X} := \{\mathbf{x}^1, \ldots, \mathbf{x}^M\} \subset \{\pm 1\}^K$ by setting $x_k^m = 1$ iff $m \in \mathcal{A}_k^{+1}$, for all $k \in [1 \ldots K]$ and $m \in [1 \ldots M]$, then statement (8) says $\mathcal{X}$ is median-saturating. Thus,

$$\begin{aligned} m(K) &= \min\{M \in \mathbb{N};\ \text{there is an independent collection of subsets } \mathcal{A}_1^{+1}, \ldots, \mathcal{A}_K^{+1} \subseteq [1 \ldots M]\} \\ &\underset{(*)}{=} \min\left\{M \in \mathbb{N};\ K \leq \binom{M-1}{\lfloor \frac{M}{2} \rfloor - 1}\right\},\end{aligned}$$

where $(*)$ is by a result proved independently by Kleitman and Spencer [10] and Katona [9].

(c) The asymptotic growth rate of the central binomial coefficient is

$$\binom{2n}{n} = \left(\sqrt{\frac{2}{\pi}} + o(1)\right) \frac{4^n}{\sqrt{2n+1}}. \tag{9}$$

If $M := 2n + 1$, then $M - 1 = 2n$ and $\lfloor \frac{M}{2} \rfloor = n$. Thus, we have

$$\begin{aligned} f(M) &:= \binom{M-1}{\lfloor \frac{M}{2} \rfloor - 1} = \binom{2n}{n-1} = \frac{n}{n+1}\binom{2n}{n} \\ &\underset{(*)}{=} (1 + o(1)) \cdot \left(\sqrt{\frac{2}{\pi}} + o(1)\right) \frac{4^n}{\sqrt{2n+1}} \underset{(\dagger)}{=} \left(\sqrt{\frac{2}{\pi}} + o(1)\right) \cdot \frac{2^{M-1}}{\sqrt{M}}.\end{aligned} \tag{10}$$

Here $(*)$ uses Eq. (9) and the fact that $\frac{n}{n+1} = 1 + o(1)$, while $(\dagger)$ is because $(x + o(1)) \cdot (y + o(1)) = xy + o(1)$, for any constants $x, y \in \mathbb{R}$.

Now, if $M = m(K)$, then part (a) says that $f(M-1) < K \leq f(M)$. Substituting Eq. (10) into both the lower and upper bounds, multiplying by $\sqrt{M}$, and taking the $\log_2$ of the three resulting expressions, we obtain

$$\begin{aligned} \log_2\left(\sqrt{\frac{2}{\pi}} + o(1)\right) + (M-2) + \frac{1}{2}(\log_2(M) - \log_2(M-1)) &\leq \log_2(K) + \frac{1}{2}\log_2(M) \\ &\leq \log_2\left(\sqrt{\frac{2}{\pi}} + o(1)\right) + (M-1).\end{aligned} \tag{11}$$

But $(\log_2(M) - \log_2(M-1)) = \mathcal{O}(\log'(M)) = \mathcal{O}(\frac{1}{M})$. Thus, we can reorganize (11) to obtain

$$M - 1 + \mathcal{O}(1/M) \leq \log_2(K) + \frac{1}{2}\log_2(M) + 1 - \log_2\left(\sqrt{\frac{2}{\pi}} + o(1)\right) \leq M.$$

Thus, $M = \log_2(K) + \frac{1}{2}\log_2(M) + \mathcal{O}(1)$. Substituting this expression into itself, we get

$$\begin{aligned} M &= \log_2(K) + \frac{1}{2}\log_2\left(\log_2(K) + \frac{1}{2}\log_2(M) + \mathcal{O}(1)\right) + \mathcal{O}(1) \\ &= \log_2(K) + \frac{1}{2}\log_2(\log_2(K)) + \mathcal{O}\left(\frac{\frac{1}{2}\log_2(M) + \mathcal{O}(1)}{\log_2(K)}\right) + \mathcal{O}(1) \\ &= \log_2(K) + \frac{1}{2}\log_2(\log_2(K)) + \mathcal{O}(1),\end{aligned}$$

as desired.

(b) Fix $K \in \mathbb{N}$. Let $m := \min\{|\mathcal{X}|;\ \mathcal{X} \subseteq \{\pm 1\}^K \text{ is median-saturating}\}$, and let $L := \lceil \log_2(K) \rceil$.

"$m \leq 2L + 2$" Let $\mathcal{K} := [0 \ldots K-1]$. For any $\ell \in [0 \ldots L]$ and any $k \in \mathcal{K}$, let $\beta_\ell(k) \in \{0, 1\}$ be the ℓth digit in the binary expansion of the number k (so that $k = \sum_{\ell=0}^{L-1} \beta_\ell(k) \cdot 2^\ell$). Then define $\mathbf{x}^\ell \in \{\pm 1\}^{\mathcal{K}}$ by $x_k^\ell := (-1)^{\beta_\ell(k)}$ for all $k \in \mathcal{K}$.

(For example, if $K = 8$, then $L = 3$, and we have: $\mathbf{x}^0 := (1, -1, 1, -1, 1, -1, 1, -1)$, $\mathbf{x}^1 := (1, 1, -1, -1, 1, 1, -1, -1)$, $\mathbf{x}^2 := (1, 1, 1, 1, -1, -1, -1, -1)$, and $\mathbf{x}^3 := (1, 1, 1, 1, 1, 1, 1, 1)$.) Now let $\mathcal{X} := \{\pm\mathbf{x}^\ell\}_{\ell=0}^L$.

Claim 1. $\mathcal{W}_2(\mathcal{X}) = \emptyset$.

Proof. Let $j, k \in \mathcal{K}$ be distinct. Then j and k must have different binary expansions. Thus, there exists some $\ell \in [0 \dots L-1]$ such that $\beta_\ell(k) \neq \beta_\ell(j)$, and hence $x_k^\ell \neq x_j^\ell$. Thus, $\pm\mathbf{x}^\ell$ realize the $\{j, k\}$-words $(-1, 1)$ and $(1, -1)$. On the other hand $\mathbf{x}^L = \mathbf{1}$, so that $\pm\mathbf{x}^L = \pm\mathbf{1}$ realize the $\{j, k\}$-words $(1, 1)$ and $(-1, -1)$. Thus, none of the four possible $\{j, k\}$-words is $\mathcal{X}$-forbidden. This holds for all $j, k \in \mathcal{K}$; hence $\mathcal{W}_2(\mathcal{X}) = \emptyset$. $\diamond$ claim 1

Proposition 1.1(b) and Claim 1 imply that $\mathcal{X}$ is median-saturating. Clearly, $|\mathcal{X}| = 2L + 2$.

"$m \geq L + 1$" Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be median-saturating. Define a function $\beta : \mathcal{K} \times \{\pm 1\} \longrightarrow \{\pm 1\}^{\mathcal{X}}$ as follows: for any $k \in \mathcal{K}$, $a \in \{\pm 1\}$, and $\mathbf{x} \in \mathcal{X}$, let $\beta(k, a)_{\mathbf{x}} := a \cdot x_k$.

Claim 2. *β is injective.*

Proof. Let $(j, a) \in \mathcal{K} \times \{\pm 1\}$ and $(k, b) \in \mathcal{K} \times \{\pm 1\}$ be distinct. We must show that $\beta(j, a) \neq \beta(k, b)$.
If $j = k$ but $a \neq b$, then $\beta(j, a) = -\beta(k, b)$; thus, $\beta(j, a) \neq \beta(k, b)$.
Now suppose $j \neq k$ and $a = b$. Proposition 1.1(b) says $\mathcal{W}_2(\mathcal{X}) = \emptyset$. Thus, there exists $\mathbf{x} \in \mathcal{X}$ with $(x_j, x_k) = (1, -1)$ Thus, $\beta(j, a)_{\mathbf{x}} = a = b \neq -b = \beta(k, b)_{\mathbf{x}}$, so $\beta(j, a) \neq \beta(k, b)$.
Finally, suppose $j \neq k$ and $a = -b$. Proposition 1.1(b) says $\mathcal{W}_2(\mathcal{X}) = \emptyset$. Thus, there exists $\mathbf{x} \in \mathcal{X}$ with $(x_j, x_k) = (1, 1)$ Thus, $\beta(j, a)_{\mathbf{x}} = a \neq b = \beta(k, b)_{\mathbf{x}}$, so $\beta(j, a) \neq \beta(k, b)$. $\diamond$ claim 2

Claim 2 implies that $|\{\pm 1\}^{\mathcal{X}}| \geq |\mathcal{K} \times \{\pm 1\}| = 2K$. Thus, $|\mathcal{X}| \geq \log_2(2K) = \log_2(K) + 1$.

(d) Let $M := \max\{|\mathcal{X}|;\ \mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ is minimal median-saturating$\}$.

"$M \geq K(K-1)/2$" For all distinct $j, k \in \mathcal{K}$, define $\mathbf{x}^{\{j,k\}} \in \{\pm 1\}^{\mathcal{K}}$ by $x_j^{\{j,k\}} = x_k^{\{j,k\}} = 1$, while $x_i^{\{j,k\}} = -1$ for all $i \in \mathcal{K} \setminus \{j, k\}$. Let $\mathcal{X} := \{\mathbf{x}^{\{j,k\}};\ \{j, k\} \subset \mathcal{K}\}$. Then $|\mathcal{X}| = K(K-1)/2$.

Claim 3. $\mathcal{W}_2(\mathcal{X}) = \emptyset$.

Proof. Fix $\{j, k\} \subset \mathcal{K}$. Clearly, $x_{j,k}^{\{j,k\}} = (1, 1)$. For any $i \in \mathcal{K} \setminus \{j, k\}$, we have $x_{j,k}^{\{i,k\}} = (-1, 1)$ and $x_{j,k}^{\{j,i\}} = (1, -1)$. Finally, for any $h, i \in \mathcal{K} \setminus \{j, k\}$, we have $x_{j,k}^{\{h,i\}} = (-1, -1)$ (recall $K \geq 4$). Thus, all four words in $\{\pm 1\}^{\{j,k\}}$ are $\mathcal{X}$-admissible. This holds for any $\{j, k\} \subset \mathcal{K}$. Thus, $\mathcal{W}_2(\mathcal{X}) = \emptyset$. $\diamond$ claim 3

Proposition 1.1(b) and Claim 3 imply that $\mathcal{X}$ is median-saturating. But if we remove any element from $\mathcal{X}$, then this argument breaks down. For example, let $\mathcal{X}' := \mathcal{X} \setminus \{\mathbf{x}^{\{j,k\}}\}$ for some $\{j, k\} \subset \mathcal{K}$. Then $\mathbf{x}_{j,k} \neq (1, 1)$ for all $\mathbf{x} \in \mathcal{X}'$ Thus, $\mathcal{W}_2(\mathcal{X}') \neq \emptyset$, so Proposition 1.1(b) implies that $\mathcal{X}'$ is *not* median-saturating. Thus, $\mathcal{X}$ is *minimal* median-saturating; thus, $M \geq |\mathcal{X}| = K(K-1)/2$.

"$M \leq 2K(K-1)$" Suppose $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ is minimal median-saturating. For every $\mathbf{x} \in \mathcal{X}$, let $\mathcal{W}(\mathbf{x}) := \mathcal{W}_2(\mathcal{X} \setminus \{\mathbf{x}\})$.

Claim 4. (a) *For all $\mathbf{x} \in \mathcal{X}$, we have $\mathcal{W}(\mathbf{x}) \neq \emptyset$.*
(b) *For all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, the sets $\mathcal{W}(\mathbf{x})$ and $\mathcal{W}(\mathbf{y})$ are disjoint.*

Proof. (a) For every $\mathbf{x} \in \mathcal{X}$, the set $\mathcal{X} \setminus \{\mathbf{x}\}$ is *not* median-saturating, so Proposition 1.1(b) says $\mathcal{W}_2(\mathcal{X} \setminus \{\mathbf{x}\}) \neq \emptyset$.
(b) Let $\mathbf{w} \in \mathcal{W}(\mathbf{x})$. Then $\mathbf{w} \not\sqsubset \mathbf{y}$ for any $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$. However, $\mathbf{w} \notin \mathcal{W}_2(\mathcal{X})$; hence we must have $\mathbf{w} \sqsubset \mathbf{x}$. If $\mathbf{w} \in \mathcal{W}(\mathbf{y})$ for some other $\mathbf{y} \in \mathcal{X}$, then the same argument shows that $\mathbf{w} \sqsubset \mathbf{y}$ but $\mathbf{w} \not\sqsubset \mathbf{x}$. Contradiction. $\diamond$ claim 4

Let $\mathcal{W}_2$ be the set of all words of length 2. Then $|\mathcal{W}_2| = 4\binom{K}{2} = 2K(K-1)$. Claim 4 shows that

$$|\mathcal{W}_2| \underset{(b)}{\geq} \sum_{\mathbf{x}\in\mathcal{X}} |\mathcal{W}(\mathbf{x})| \underset{(a)}{\geq} \sum_{\mathbf{x}\in\mathcal{X}} 1 = |\mathcal{X}|.$$

Thus, $|\mathcal{X}| \leq 2K(K-1)$. $\square$

Lemma A.1. *Let $\mathcal{S} \subset \mathbb{R}^{\mathcal{K}}$ be an affine subspace of dimension $D \leq K$. Then $|\mathcal{S} \cap \{\pm 1\}^{\mathcal{K}}| \leq 2^D$.*

Proof. Suppose $\mathcal{K} = [1 \dots K]$, and identify $\mathbb{R}^{\mathcal{K}}$ with $\mathbb{R}^D \times \mathbb{R}^{K-D}$ in the obvious way. If $\dim(\mathcal{S}) = D$, then there exists some affine function $\phi : \mathbb{R}^D \longrightarrow \mathbb{R}^{K-D}$ such that (after some permutation of $\mathcal{K}$), we have $\mathcal{S} = \{(\mathbf{r}, \phi(\mathbf{r}));\ \mathbf{r} \in \mathbb{R}^D\}$. This means that $\mathcal{S} \cap \{\pm 1\}^{\mathcal{K}} = \{(\mathbf{x}, \phi(\mathbf{x}));\ \mathbf{x} \in \{\pm 1\}^D$ and $\phi(\mathbf{x}) \in \{\pm 1\}^{K-D}\}$. Thus, $|\mathcal{S} \cap \{\pm 1\}^{\mathcal{K}}| \leq |\{\pm 1\}^D| = 2^D$. $\square$

Proof of Proposition 2.4. (a) Let $M_0 := \max\{|\mathcal{X}|;\ \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is not McGarvey$\}$.

"$M_0 \geq \frac{3}{4}2^K$" follows immediately from Example 2.5. To see "$M_0 \leq \frac{3}{4}2^K$", suppose $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ is *not* McGarvey. Then Theorem 1.3(b3) says there exists nonzero $\mathbf{z} \in \mathbb{R}^{\mathcal{K}}$, such that $\mathbf{z} \bullet \mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathcal{X}$. Let $\mathcal{Y}_+ := \{\mathbf{y} \in \{\pm 1\}^{\mathcal{K}};\ \mathbf{z} \bullet \mathbf{y} > 0\}$, let

$\mathcal{Y}_- := \{\mathbf{y} \in \{\pm 1\}^{\mathcal{K}}; \mathbf{z} \bullet \mathbf{y} < 0\}$, and let $\mathcal{Y}_0 := \{\mathbf{y} \in \{\pm 1\}^{\mathcal{K}}; \mathbf{z} \bullet \mathbf{y} = 0\}$. Now, $|\mathcal{Y}_-| = |\mathcal{Y}_+|$ (because these sets are images of one another under negation). Thus,

$$|\mathcal{Y}_-| = \frac{1}{2}|\{\pm 1\}^{\mathcal{K}} \setminus \mathcal{Y}_0| = \frac{1}{2}(2^K - |\mathcal{Y}_0|) = 2^{K-1} - \frac{1}{2}|\mathcal{Y}_0|. \tag{12}$$

$$\text{Also,} \quad \mathcal{X} \subseteq \mathcal{Y}_- \sqcup \mathcal{Y}_0.$$

$$\text{Thus,} \quad |\mathcal{X}| \le |\mathcal{Y}_- \sqcup \mathcal{Y}_0| = |\mathcal{Y}_-| + |\mathcal{Y}_0| \underset{(\dagger)}{=} 2^{K-1} - \frac{1}{2}|\mathcal{Y}_0| + |\mathcal{Y}_0| = 2^{K-1} + \frac{1}{2}|\mathcal{Y}_0|$$
$$\underset{(*)}{\le} 2^{K-1} + \frac{1}{2}2^{K-1} = \frac{3}{4}2^K,$$

as claimed. Here, $(\dagger)$ is by Eq. (12), and $(*)$ is because $|\mathcal{Y}_0| \le 2^{K-1}$ by Lemma A.1.

(b) Let $M_1 := \max\{|\mathcal{X}|; \mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ is not median-saturating$\}$.

"$M_1 \ge \frac{3}{4}2^K$" follows immediately from Example 2.5. To see "$M_1 \le \frac{3}{4}2^K$", observe that $\{\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}; \mathcal{X}$ is not median-saturating$\} \subseteq \{\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}; \mathcal{X}$ is not McGarvey$\}$ (because McGarvey implies median-saturating). Thus, $M_1 \le M_0$, and we have already verified that $M_0 \le \frac{3}{4}2^K$. □

Proof of Proposition 3.1. (a) Let $\mathbf{z} := \sum_{\mathbf{x}\in\mathcal{X}} \mathbf{x}$. Then $\gamma(\mathbf{z}) = \mathbf{z}$ for all $\gamma \in \Gamma_{\mathcal{X}}$; hence $\mathbf{z} \in \mathrm{Fix}(\Gamma_{\mathcal{X}})$, which means $\mathbf{z} = \mathbf{0}$ (by hypothesis). Thus, $\frac{1}{|\mathcal{X}|}\sum_{\mathbf{x}\in\mathcal{X}} \mathbf{x} = \mathbf{0}$, so Theorem 1.3(b4) says $\mathcal{X}$ is McGarvey.

(b) If $-\mathcal{X} = \mathcal{X}$, then $-\mathbf{I} \in \Gamma_{\mathcal{X}}$. Thus, for any $\mathbf{r} \in \mathrm{Fix}(\Gamma_{\mathcal{X}})$, we have $-\mathbf{r} = \mathbf{r}$, which means $\mathbf{r} = \mathbf{0}$. Thus, $\mathrm{Fix}(\Gamma_{\mathcal{X}}) = \{\mathbf{0}\}$. Thus, part (a) says $\mathcal{X}$ is McGarvey. □

Proof of Lemma 3.2. Let $\mathcal{Y} := \{\mathbf{x} - \mathbf{y}; \mathbf{x}, \mathbf{y} \in \mathcal{X}\}$. For all $j \in \mathcal{K}$, let $\mathbf{e}^j := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the '1' appears in the jth coordinate. If $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ are such that $x_j \ne y_j$, but $x_k = y_k$ for all $k \in \mathcal{K} \setminus \{j\}$, then $\mathbf{x} - \mathbf{y} = \pm\mathbf{e}^j$. Thus, by hypothesis, $\mathcal{Y}$ contains $\{\pm\mathbf{e}^j\}_{j\in\mathcal{K}}$. Thus, $\mathrm{span}(\mathcal{Y}) = \mathbb{R}^{\mathcal{K}}$. Thus, $\mathrm{int}[\mathrm{conv}(\mathcal{X})] \ne \emptyset$. □

Proof of Example 3.4. For any $\mathcal{M} \subseteq [1 \ldots L]$, recall that $\mathcal{A}_{\mathcal{M}} := \prod_{m\in\mathcal{M}} \mathcal{A}_m$ and $\mathcal{A}_{\mathcal{M}^\complement} := \prod_{n\notin\mathcal{M}} \mathcal{A}_n$. Fix $\mathbf{c} \in \mathcal{A}$ and let "$>$" be an arbitrary reference order on $\mathcal{A}$. Define $\widetilde{\mathcal{K}} := \{(\mathcal{M}, \mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}}); \emptyset \ne \mathcal{M} \subseteq [1 \ldots L]$ and $\mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}} \in \mathcal{A}_{\mathcal{M}}$ such that $(\mathbf{a}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement}) > (\mathbf{b}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement})$ and $a_m \ne b_m$ for all $m \in \mathcal{M}\}$. There is a bijection between $\{\pm 1\}^{\widetilde{\mathcal{K}}}$ and the separable tournaments on $\mathcal{A}$. If "$\succ$" is any separable tournament on $\mathcal{A}$, then we can define an element $\mathbf{x}^{\succ} \in \{\pm 1\}^{\widetilde{\mathcal{K}}}$ by setting $x^{\succ}_{(\mathcal{M},\mathbf{a}_{\mathcal{M}},\mathbf{b}_{\mathcal{M}})} = 1$ if $(\mathbf{a}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement}) \succ (\mathbf{b}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement})$, while $x^{\succ}_{(\mathcal{M},\mathbf{a}_{\mathcal{M}},\mathbf{b}_{\mathcal{M}})} = -1$ if $(\mathbf{a}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement}) \prec (\mathbf{b}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement})$. Every element of $\{\pm 1\}^{\widetilde{\mathcal{K}}}$ corresponds to a unique separable tournament in this way.

Let $\widetilde{\mathcal{X}}^{\mathrm{add}}_{\mathcal{A}} \subset \{\pm 1\}^{\widetilde{\mathcal{K}}}$ be the space of additively separable preference orders. We will show that $\mathrm{span}(\widetilde{\mathcal{X}}^{\mathrm{add}}_{\mathcal{A}}) = \mathbb{R}^{\widetilde{\mathcal{K}}}$. For all $\ell \in [1 \ldots L]$, let $u_\ell : \mathcal{A}_\ell \longrightarrow \mathbb{R}$ be a 'utility function'. For any $\mathbf{a} \in \mathcal{A}$, define $U(\mathbf{a}) := u_1(a_1) + \cdots + u_L(a_L)$. For a generic choice of functions $u_1, \ldots, u_L$, we will have $U(\mathbf{a}) \ne U(\mathbf{b})$ whenever $\mathbf{a} \ne \mathbf{b}$. Then we obtain an additively separable preference order "$\underset{U}{\succ}$" on $\mathcal{A}$ by setting $\mathbf{a} \underset{U}{\succ} \mathbf{b}$ if and only if $U(\mathbf{a}) > U(\mathbf{b})$.

Let $(\mathcal{M}, \mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}}) \in \widetilde{\mathcal{K}}$. Define $\mathbf{a} := (\mathbf{a}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement})$ and $\mathbf{b} := (\mathbf{b}_{\mathcal{M}}, \mathbf{c}_{\mathcal{M}^\complement})$. Construct $u_1, \ldots, u_L$ such that:

- $U(\mathbf{a}) = U(\mathbf{b}) \ne U(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{A} \setminus \{\mathbf{a}, \mathbf{b}\}$; and
- $U(\mathbf{d}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement}) \ne U(\mathbf{e}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement})$ for all $(\mathcal{N}, \mathbf{d}_{\mathcal{N}}, \mathbf{e}_{\mathcal{N}}) \in \widetilde{\mathcal{K}} \setminus \{(\mathcal{M}, \mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}})\}$.

Let

$$0 < \epsilon < \frac{1}{2}\min\{|U(\mathbf{d}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement}) - U(\mathbf{e}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement})|; (\mathcal{N}, \mathbf{d}_{\mathcal{N}}, \mathbf{e}_{\mathcal{N}}) \in \widetilde{\mathcal{K}} \setminus \{(\mathcal{M}, \mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}})\}\}.$$

Without loss of generality, suppose $1 \in \mathcal{M}$. Define utility functions $u_1^+ : \mathcal{A}_1 \longrightarrow \mathbb{R}$ and $u_1^- : \mathcal{A}_1 \longrightarrow \mathbb{R}$ by setting $u_1^+(a_1) := u_1(a_1) + \epsilon$ and $u_1^-(a_1) := u_1(a_1) - \epsilon$, whereas $u_1^{\pm}(d) := u_1(d)$ for all $d \in \mathcal{A}_1 \setminus \{a_1\}$. (In particular, $u^{\pm}(b_1) = u(b_1)$; recall that $b_1 \ne a_1$ by definition of $\widetilde{\mathcal{K}}$.) Now define $U^{\pm}(a_1, \ldots, a_L) := u_1^{\pm}(d_1) + u_2(d_2) \cdots + u_L(d_L)$ for all $(d_1, \ldots, d_L) \in \mathcal{A}$. Then

- $U^+(\mathbf{d}) \ne U^+(\mathbf{e})$ and $U^-(\mathbf{d}) \ne U^-(\mathbf{e})$, for all $\mathbf{d} \ne \mathbf{e} \in \mathcal{A}$;
- $U^+(\mathbf{a}) > U^+(\mathbf{b})$ and $U^-(\mathbf{a}) < U^-(\mathbf{b})$; and
- for all $(\mathcal{N}, \mathbf{d}_{\mathcal{N}}, \mathbf{e}_{\mathcal{N}}) \in \widetilde{\mathcal{K}} \setminus \{(\mathcal{M}, \mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}})\}$, we have

$$(U^+(\mathbf{d}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement}) > U^+(\mathbf{e}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement})) \iff (U^-(\mathbf{d}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement}) > U^-(\mathbf{e}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement})).$$

We thus obtain well-defined additively separable preference orders "$\underset{U^+}{\succ}$" and "$\underset{U^-}{\succ}$", such that:

- $\mathbf{a} \underset{U^+}{\succ} \mathbf{b}$, while $\mathbf{a} \underset{U^-}{\prec} \mathbf{b}$; and

- for all $(\mathcal{N}, \mathbf{d}_{\mathcal{N}}, \mathbf{e}_{\mathcal{N}}) \in \widetilde{\mathcal{K}} \setminus \{(\mathcal{M}, \mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}})\}$, we have

$$((\mathbf{d}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement}) \underset{U^+}{\succ} (\mathbf{e}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement})) \iff ((\mathbf{d}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement}) \underset{U^-}{\succ} (\mathbf{e}_{\mathcal{N}}, \mathbf{c}_{\mathcal{N}^\complement})).$$

Thus, if $\mathbf{x}^+, \mathbf{x}^- \in \widetilde{\mathcal{X}}^{\mathrm{add}}_{\mathcal{A}}$ are the elements corresponding to "$\underset{U^+}{\succ}$" and "$\underset{U^-}{\succ}$", then $x^-_{(\mathcal{M},\mathbf{a}_{\mathcal{M}},\mathbf{b}_{\mathcal{M}})} = -x^+_{(\mathcal{M},\mathbf{a}_{\mathcal{M}},\mathbf{b}_{\mathcal{M}})}$, whereas $x^+_k = x^-_k$ for all other $k \in \widetilde{\mathcal{K}}$.

We can do this for any $(\mathcal{M}, \mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}}) \in \widetilde{\mathcal{K}}$; thus, Lemma 3.2 implies that $\mathrm{span}(\widetilde{\mathcal{X}}^{\mathrm{add}}_{\mathcal{A}}) = \mathbb{R}^{\widetilde{\mathcal{K}}}$. Since $-\widetilde{\mathcal{X}}^{\mathrm{add}}_{\mathcal{A}} = \widetilde{\mathcal{X}}^{\mathrm{add}}_{\mathcal{A}}$, Proposition 3.1(b) says $\mathrm{maj}(\widetilde{\mathcal{X}}^{\mathrm{add}}_{\mathcal{A}}) = \{\pm 1\}^{\widetilde{\mathcal{K}}}$. □

Proof of Example 3.5. We must show $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$. Let $\epsilon := \min\{|j_d - k_d|; \mathbf{j}, \mathbf{k} \in \mathcal{K}, d \in [1 \ldots D], \text{ and } j_d \neq k_d\}$. By replacing $\mathcal{K}$ with $\mathcal{K}' := \frac{1}{\epsilon}\mathcal{K}$ if necessary, we can assume without loss of generality that $\epsilon \geq 1$.

Let "$\underset{\mathrm{lex}}{\prec}$" be the lexicographical order on $\mathcal{K}$. That is: $\mathbf{j} \underset{\mathrm{lex}}{\prec} \mathbf{k}$ if there is some $C \in [1 \ldots D]$ such that $j_d = k_d$ for all $d \in [1 \ldots C-1]$, while $j_C < k_C$. This is a well-ordering of $\mathcal{K}$. For all $d \in [2 \ldots D]$, let $M_d > (D-1)\cdot(\max\{k_d; \mathbf{k} \in \mathcal{K}\} - \min\{k_d; \mathbf{k} \in \mathcal{K}\})$. (The max and min are well defined and finite because $\mathcal{K}$ is finite.) Define $\mathbf{r} := (1, \frac{1}{M_2}, \frac{1}{M_2 M_3}, \frac{1}{M_2 M_3 M_4}, \ldots, \frac{1}{M_2 \cdots M_D})$. Then for all $\mathbf{i}, \mathbf{j} \in \mathcal{K}$, we have $(\mathbf{i} \underset{\mathrm{lex}}{\prec} \mathbf{j}) \iff (\mathbf{r} \bullet \mathbf{i} < \mathbf{r} \bullet \mathbf{j})$. Thus, for any $\mathbf{j} \in \mathcal{K}$, if $q(\mathbf{j}) := \mathbf{r} \bullet \mathbf{j}$, then $\mathcal{H}^{\mathbf{r}}_{q(\mathbf{j})} = \{\mathbf{i} \in \mathcal{K}; \mathbf{i} \underset{\mathrm{lex}}{\preceq} \mathbf{j}\}$. Thus, if $\mathbf{k}$ is lexicographically minimal in the set $\{\mathbf{k} \in \mathcal{K}; \mathbf{j} \underset{\mathrm{lex}}{\prec} \mathbf{k}\}$, then $\mathbf{x}^{\mathbf{r}}_{q(\mathbf{j})}$ and $\mathbf{x}^{\mathbf{r}}_{q(\mathbf{k})}$ differ only in coordinate $\mathbf{k}$. Now apply Lemma 3.2. □

Proof of Proposition 3.6. Let $A := |\mathcal{A}|$, and suppose without loss of generality that $\mathcal{A} := [1 \ldots A]$ and $\mathcal{K} := \{(a, b); a, b \in \mathcal{A} \text{ and } a < b\}$. For any $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ and any $a < b \in \mathcal{A}$, we will abuse notation by defining

$$r_{b,a} := -r_{a,b}. \tag{13}$$

"$\Longrightarrow$" (by contrapositive). Suppose every element of $\mathcal{T}$ has an Eulerian circuit. Let $\mathcal{V} := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}}; \sum_{b=2}^{A} r_{1,b} = 0\}$. Then $\mathcal{V}$ is a linear subspace of $\mathbb{R}^{\mathcal{K}}$, with $\dim(\mathcal{V}) = K - 1$.

Now, for all $\mathbf{x} \in \mathcal{X}_{\mathcal{T}}$, we have $\#\mathrm{In}_1(\mathbf{T_x}) = \#\mathrm{Out}_1(\mathbf{T_x})$, which means $\sum_{b=2}^{A} x_{1,b} = 0$, so $\mathbf{x} \in \mathcal{V}$. Thus, $\mathcal{X}_{\mathcal{T}} \subset \mathcal{V}$. Thus, $\mathrm{conv}(\mathcal{X}_{\mathcal{T}}) \subset \mathcal{V}$. Thus, $\mathrm{int}[\mathrm{conv}(\mathcal{X}_{\mathcal{T}})] = \emptyset$. Thus, $\mathcal{X}_{\mathcal{T}}$ cannot be McGarvey.

"$\Longleftarrow$" We will prove $\mathcal{X}$ is McGarvey using Proposition 3.1(a). For any $\pi \in \Pi_{\mathcal{A}}$, define linear transformation $\pi^* : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R}^{\mathcal{K}}$ as follows: for any $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ and $a < b \in \mathcal{A}$, we define $\pi^*(\mathbf{r})_{a,b} := r_{\pi(a),\pi(b)}$ (following convention (13) above). If $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ and $\pi^*(\mathbf{x}) = \mathbf{y}$, then $\pi(\mathbf{T_x}) = \mathbf{T_y}$. Thus, $\pi^*(\mathcal{X}_{\mathcal{T}}) = \mathcal{X}_{\mathcal{T}}$ because $\pi(\mathcal{T}) = \mathcal{T}$. Thus, if $\Pi^*_{\mathcal{A}} := \{\pi^*; \pi \in \Pi_{\mathcal{A}}\}$, then $\Pi^*_{\mathcal{A}} \subseteq \Gamma_{\mathcal{X}_{\mathcal{T}}}$.

Claim 1. $\mathrm{Fix}(\Pi^*_{\mathcal{A}}) = \{\mathbf{0}\}$.

Proof. Let $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ and suppose $\pi^*(\mathbf{r}) = \mathbf{r}$ for all $\pi^* \in \Pi^*_{\mathcal{A}}$; we must show that $\mathbf{r} = \mathbf{0}$. So, let $(a, b) \in \mathcal{K}$. Find $\pi \in \Pi_{\mathcal{A}}$ with $\pi(a) = b$ and $\pi(b) = a$. Then $\pi^*(\mathbf{r}) = \mathbf{r}$, because $\pi^* \in \Pi^*_{\mathcal{A}}$. Thus, $r_{a,b} = \pi^*(\mathbf{r})_{a,b} = r_{b,a} = -r_{a,b}$. Thus, $r_{a,b} = 0$. This holds for all $a, b \in \mathcal{A}$. Thus, $\mathbf{r} = \mathbf{0}$. ◇ claim 1

At this point it remains to show that $\mathrm{span}(\mathcal{X}_{\mathcal{T}}) = \mathbb{R}^{\mathcal{K}}$.

Claim 2. *Suppose $\mathcal{X}$ is not McGarvey. Then for any $a, b \in \mathcal{A}$, there exists some $\mathbf{y} \in \mathcal{X}$ with $y_{a,b} = 1$, such that $\#\mathrm{In}_a(\mathbf{T_y}) \geq \#\mathrm{In}_b(\mathbf{T_y})$ and $\#\mathrm{Out}_a(\mathbf{T_y}) \leq \#\mathrm{Out}_b(\mathbf{T_y})$.*

Proof.

Claim 2.1. *Let $\mathbf{y} \in \{\pm 1\}^{\mathcal{K}}$. Suppose that, for all $a, b \in \mathcal{A}$, if $y_{a,b} = 1$, then $\#\mathrm{In}_a(\mathbf{T_y}) < \#\mathrm{In}_b(\mathbf{T_y})$ and $\#\mathrm{Out}_a(\mathbf{T_y}) > \#\mathrm{Out}_b(\mathbf{T_y})$. Then $\mathbf{T_y}$ is a preference order.* □

Proof. Define the complete, antisymmetric relation "$\succ$" on $\mathcal{A}$ by $(a \succ b) \iff (y_{a,b} = 1)$. We must show that "$\succ$" is transitive. Define $u : \mathcal{A} \longrightarrow \mathbb{R}$ by $u(a) := \#\mathrm{Out}_a(\mathbf{T_y}) - \#\mathrm{In}_a(\mathbf{T_y})$. Then by hypothesis, for all $a, b \in \mathcal{A}$, we have: $(a \succ b) \Longrightarrow (u(a) > u(b))$. Since "$\succ$" is complete and antisymmetric, we can strengthen this to $(a \succ b) \iff (u(a) > u(b))$. Thus, u is a utility function for "$\succ$", so "$\succ$" must be a preference relation. ▽ claim 2.1

Claim 2.2. *For all $\mathbf{x} \in \mathcal{X}$, there exist $c, d \in \mathcal{A}$ such that $x_{c,d} = 1$, while $\#\mathrm{In}_c(\mathbf{T_x}) \geq \#\mathrm{In}_d(\mathbf{T_x})$ and $\#\mathrm{Out}_c(\mathbf{T_x}) \leq \#\mathrm{Out}_d(\mathbf{T_x})$.*

Proof (*By Contradiction*)**.** Suppose not. Then there exists some $\mathbf{y} \in \mathcal{X}$ satisfying the hypotheses of Claim 2.1, so that $\mathbf{T_y}$ is a preference order. By applying $\Pi^*_{\mathcal{A}}$ to $\mathbf{y}$, we can obtain *all* preference orders on $\mathcal{A}$. But $\mathcal{X}$ is $\Pi^*_{\mathcal{A}}$-invariant, so this means that $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}} \subseteq \mathcal{X}$; thus $\mathcal{X}$ is McGarvey because $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$ is McGarvey, which contradicts the hypothesis of Claim 2. ▽ claim 2.2

Now, take any $\mathbf{x} \in \mathcal{X}$, and find $c, d \in \mathcal{A}$ as in Claim 2.2. Then find $\pi \in \Pi_{\mathcal{A}}$ such that $\pi(a) = c$ and $\pi(b) = d$. Let $\mathbf{y} := \pi^*(\mathbf{x})$. Then $y_{a,b} = 1$, while $\#\mathrm{In}_a(\mathbf{T_y}) \geq \#\mathrm{In}_b(\mathbf{T_y})$ and $\#\mathrm{Out}_a(\mathbf{T_y}) \leq \#\mathrm{Out}_b(\mathbf{T_y})$, as desired. ◇ claim 2

Claim 3. *For any $\mathbf{x} \in \mathcal{X}_{\mathcal{T}}$, if $\overline{\mathbf{x}} := \sum_{\pi \in \Pi_{\mathcal{A}}} \pi^*(\mathbf{x})$, then $\overline{\mathbf{x}} = \mathbf{0}$.*

Proof. Clearly, $\overline{\mathbf{x}} \in \mathrm{Fix}(\Pi^*_{\mathcal{A}})$. Now apply Claim 1. ◇ claim 3

Recall $A = |\mathcal{A}|$. For any $a \in \mathcal{A}$, let $\Pi_{-a} \subset \Pi_{\mathcal{A}}$ be the set of permutations fixing a (effectively: the permutations of $\mathcal{A} \setminus \{a\}$), and let $\Pi^*_{-a} := \{\pi^*; \pi \in \Pi_{-a}\}$.

Claim 4. *Let* $\mathbf{x} \in \mathcal{X}_{\mathcal{T}}$, *and let* $\mathbf{r} := \frac{1}{|\Pi_{-a}|} \sum_{\pi \in \Pi_{-a}} \pi^*(\mathbf{x})$. *Let* $\overline{x}_a := \frac{1}{A-1} \sum_{b \in \mathcal{A} \setminus \{a\}} x_{a,b}$. *Then:*

(a) $r_{b,c} = 0$ *for all* $b, c \in \mathcal{A} \setminus \{a\}$.
(b) $r_{a,b} = \overline{x}_a$ *for all* $b \in \mathcal{A} \setminus \{a\}$.

Proof. (a) Let $\mathcal{A}' := \mathcal{A} \setminus \{a\}$, let $\mathcal{K}' := \{(b, c);\ b, c \in \mathcal{A}' \quad \text{and} \quad b < c\}$; then the set of all tournaments on $\mathcal{A}'$ bijectively maps to $\{\pm 1\}^{\mathcal{K}'}$ in the obvious way. If $\mathbf{x} \in \mathcal{X}_{\mathcal{T}}$, and $\mathbf{x}'$ is the projection of $\mathbf{x}$ onto $\{\pm 1\}^{\mathcal{K}'}$, then $\mathbf{x}'$ represents the tournament on $\mathcal{A}'$ obtained by deleting vertex a (and all adjoining edges) from $\mathbf{T}_{\mathbf{x}}$. Let $\mathcal{X}' := \{\mathbf{y}'; \mathbf{y} \in \mathcal{X}_{\mathcal{T}}\} \subset \{\pm 1\}^{\mathcal{K}'}$. The group Π_{-a} is isomorphic to the group $\Pi_{\mathcal{A}'}$ in the obvious way, and $\Pi^*_{\mathcal{A}'} \subseteq \Gamma_{\mathcal{X}'}$ because $\Pi^* \subseteq \Gamma_{\mathcal{X}_{\mathcal{T}}}$. Claim 3 (applied to $\Pi_{\mathcal{A}'}$ and $\mathcal{X}'$) implies that $\overline{\mathbf{x}}' := \sum_{\pi \in \Pi_{\mathcal{A}'}} \pi^*(\mathbf{x}') = \mathbf{0}'$ Thus, for all $b, c \in \mathcal{A}'$, we have

$$r_{b,c} = \frac{1}{|\Pi_{-a}|} \sum_{\pi \in \Pi_{-a}} \pi^*(\mathbf{x})_{b,c} = \frac{1}{|\Pi_{-a}|} \overline{x}'_{b,c} = 0,$$

which proves part (a). Part (b) follows because Π_{-a} acts transitively on the $(A-1)$ edges connecting to a. $\diamond$ claim 4

Claim 5. $\text{span}(\mathcal{X}_{\mathcal{T}}) = \mathbb{R}^{\mathcal{K}}$.

Proof. By hypothesis, there exists some $\mathbf{T} \in \mathcal{T}$ and some $a \in \mathcal{A}$ such that $\#\text{In}_a(\mathbf{T}) \neq \#\text{Out}_a(\mathbf{T})$. Since $\mathcal{T}$ is invariant under vertex permutations, we can permute $\mathbf{T}$ to move a to the vertices of our choice.
So, let $a \in \mathcal{A}$. Find $\mathbf{x} \in \mathcal{X}_{\mathcal{T}}$ such that $\#\text{Out}_a(\mathbf{T}_{\mathbf{x}}) \neq \#\text{In}_a(\mathbf{T}_{\mathbf{x}})$. Then $\overline{x}_a \neq 0$ in the notation of Claim 4. Define

$$\mathbf{r} := \frac{1}{|\Pi_{-a}|} \sum_{\pi \in \Pi_{-a}} \pi^*(\mathbf{x}).$$

Then Claim 4 implies that $r_{c,d} = 0$ for all $c, d \in \mathcal{A} \setminus \{a\}$, while $r_{a,c} = \overline{x}_a$ for all $c \in \mathcal{A} \setminus \{a\}$. Clearly $\mathbf{r} \in \text{span}(\mathcal{X}_{\mathcal{T}})$, because $\pi^*(\mathbf{x}) \in \mathcal{X}_{\mathcal{T}}$ for all $\pi \in \Pi_{-a}$ because $\Pi^*_{-a} \subset \Pi^*_{\mathcal{A}} \subset \Gamma_{\mathcal{X}_{\mathcal{T}}}$.
Next, let $b \in \mathcal{A} \setminus \{a\}$, find $\pi \in \Pi_{\mathcal{A}}$ be such that $\pi(b) = a$, and let $\mathbf{x}' := \pi^*(\mathbf{x}) \in \mathcal{X}_{\mathcal{T}}$. Then $\overline{x}'_b = \overline{x}_a \neq 0$ in the notation of Claim 4. Thus, if we define

$$\mathbf{r}' := \frac{1}{|\Pi_{-b}|} \sum_{\pi \in \Pi_{-b}} \pi^*(\mathbf{x}'),$$

then Claim 4 implies that $r'_{c,d} = 0$ for all $c, d \in \mathcal{A} \setminus \{b\}$, while $r'_{b,c} = \overline{x}'_b$ for all $c \in \mathcal{A} \setminus \{b\}$.
Now, let $\mathbf{y}$ be as in Claim 2. Let $\Pi_{-a,b} \subset \Pi_{\mathcal{A}}$ be the group of all permutations of $\mathcal{A}$ which fix both a and b, and define

$$\mathbf{s} := \frac{1}{|\Pi_{-a,b}|} \sum_{\pi \in \Pi_{-a,b}} \pi^*(\mathbf{y}), \qquad \overline{y}_a := \frac{1}{A-2} \sum_{c \in \mathcal{A} \setminus \{a,b\}} y_{a,c}, \quad \text{and} \quad \overline{y}_b := \frac{1}{A-2} \sum_{c \in \mathcal{A} \setminus \{a,b\}} y_{b,c}.$$

Then by an argument similar to Claim 4, we have $s_{a,c} = \overline{y}_a$ and $s_{b,c} = \overline{y}_b$ for all $c \in \mathcal{A} \setminus \{a, b\}$, and $s_{c,d} = 0$ for all $c, d \in \mathcal{A} \setminus \{a, b\}$, while $s_{a,b} = 1$. Thus, if we define

$$\mathbf{z}^{a,b} := \mathbf{s} - \frac{\overline{y}_a}{\overline{x}_a} \mathbf{r} - \frac{\overline{y}_b}{\overline{x}'_b} \mathbf{r}',$$

then $z^{a,b}_{c,d} = 0$ whenever either $c \neq a$ or $d \neq b$. However,

$$z^{a,b}_{a,b} = s_{a,b} - \frac{\overline{y}_a}{\overline{x}_a} r_{a,b} - \frac{\overline{y}_b}{\overline{x}'_b} r'_{a,b} \underset{(\diamond)}{=} s_{a,b} - \frac{\overline{y}_a}{\overline{x}_a} r_{a,b} + \frac{\overline{y}_b}{\overline{x}'_b} r'_{b,a} \underset{(*)}{=} 1 - \overline{y}_a + \overline{y}_b \underset{(\dagger)}{\geq} 1.$$

Here, $(\diamond)$ is by convention (13), and $(*)$ is because $r_{a,b} = \overline{x}_a$ and $r'_{b,a} = \overline{x}'_b$. Meanwhile, $(\dagger)$ is because $\overline{y}_b - \overline{y}_a \geq 0$ because

$$\begin{aligned} \overline{y}_a &= \frac{\#\text{Out}_a(\mathbf{y}) - \#\text{In}_a(\mathbf{y}) - 1}{A-2} \underset{(\dagger)}{\leq} \frac{\#\text{Out}_b(\mathbf{y}) - \#\text{In}_b(\mathbf{y}) - 1}{A-2} \\ &\leq \frac{\#\text{Out}_b(\mathbf{y}) - \#\text{In}_b(\mathbf{y}) + 1}{A-2} = \overline{y}_b, \end{aligned}$$

where $(\dagger)$ is by the inequalities in Claim 2.
Clearly, $\mathbf{z}^{a,b} \in \text{span}(\mathcal{X}_{\mathcal{T}})$. We can do this for any $a \neq b \in \mathcal{A}$. The collection $\{\mathbf{z}^{a,b}; a \neq b \in \mathcal{A}\}$ clearly spans $\mathbb{R}^{\mathcal{K}}$. Thus, $\text{span}(\mathcal{X}_{\mathcal{T}}) = \mathbb{R}^{\mathcal{K}}$. $\diamond$ claim 5

Proposition 3.1(a), plus Claims 1 and 5, imply that $\mathcal{X}_{\mathcal{T}}$ is McGarvey. $\square$

Proof of Proposition 3.7. "$\Longrightarrow$" (by contrapositive) Suppose there do *not* exist $r < 0 < t \in \mathbb{R}$ such that $r\mathbf{1}, t\mathbf{1} \in \text{conv}(\mathcal{X})$. Then $\mathbf{0} \notin \text{int}[\text{conv}(\mathcal{X})]$. Thus, Theorem 1.3(b2) says $\mathcal{X}$ is not McGarvey.

Likewise, if $\text{span}(\mathcal{X}) \neq \mathbb{R}^{\mathcal{K}}$, then Theorem 1.3(b4) says $\mathcal{X}$ is not McGarvey.

"$\Longleftarrow$" Let $\mathbf{y} := \frac{1}{|\mathcal{X}|} \sum_{\mathbf{x} \in \mathcal{X}} \mathbf{x}$. Then $\mathbf{y} \in \text{int}[\text{conv}(\mathcal{X})]$ (same argument as Theorem 1.3 "(b4)$\Longrightarrow$(b2)"). However, $\mathbf{y} \in \text{Fix}(\Gamma_{\mathcal{X}})$, as in part (a). Thus, $\mathbf{y} = s\mathbf{1}$ for some $s \in \mathbb{R}$ (by hypothesis). If $s = 0$, then $\mathbf{y} = \mathbf{0}$, so Theorem 1.3(b4) says $\mathcal{X}$ is McGarvey. So suppose $s \neq 0$.

By hypothesis, there exist $r < 0 < t \in \mathbb{R}$ such that $r\mathbf{1}, t\mathbf{1} \in \text{conv}(\mathcal{X})$. If $s < 0$, then $\mathbf{0} = (\frac{-s}{t-s})t\mathbf{1} + (\frac{t}{t-s})\mathbf{y}$ (a strictly positive convex combination), so Theorem 1.3(b4) says $\mathcal{X}$ is McGarvey. If $s > 0$, then $\mathbf{0} = (\frac{s}{s-r})r\mathbf{1} + (\frac{-r}{s-r})\mathbf{y}$, so again Theorem 1.3(b4) says $\mathcal{X}$ is McGarvey. $\square$

Proof of Corollary 3.8. "$\Longrightarrow$" (by contrapositive) Suppose there does *not* exist any $\mathbf{x} \in \mathcal{X}$ with $\#(\mathbf{x}) < K/2$. Then $\#(\mathbf{x}) \geq K/2$ for all $\mathbf{x} \in \mathcal{X}$. This means $\sum_{k \in \mathcal{K}} x_k \geq 0$ for all $\mathbf{x} \in \mathcal{X}$ —i.e. $\mathbf{1} \bullet \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathcal{X}$. Thus, Theorem 1.3(b3) says $\mathcal{X}$ is not McGarvey.

Similarly, if $\#(\mathbf{y}) \leq K/2$ for all $\mathbf{y} \in \mathcal{X}$, then $\mathcal{X}$ cannot be McGarvey.

"$\Longleftarrow$" First note that $\text{Fix}(\Pi_{\mathcal{X}}) \subseteq \mathbb{R}\mathbf{1}$. To see this, let $\mathbf{r} \in \text{Fix}(\Pi_{\mathcal{X}})$; then $\pi(\mathbf{r}) = \mathbf{r}$ for all $\pi \in \Pi_{\mathcal{X}}$. If $\Pi_{\mathcal{X}}$ is transitive, then all coordinates of $\mathbf{r}$ must be equal; hence $\mathbf{r} \in \mathbb{R}\mathbf{1}$.

By hypothesis, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $\#(\mathbf{x}) < K/2 < \#(\mathbf{y})$. Observe that $\#[\pi(\mathbf{x})] = \#(\mathbf{x})$ and $\#[\pi(\mathbf{y})] = \#(\mathbf{y})$ for all $\pi \in \Pi_{\mathcal{X}}$. Let

$$\mathbf{x}^* := \frac{1}{|\Pi_{\mathcal{X}}|} \sum_{\pi \in \Pi_{\mathcal{X}}} \pi(\mathbf{x}) \quad \text{and} \quad \mathbf{y}^* := \frac{1}{|\Pi_{\mathcal{X}}|} \sum_{\pi \in \Pi_{\mathcal{X}}} \pi(\mathbf{y}).$$

Then $\mathbf{x}^*, \mathbf{y}^* \in \text{Fix}(\Pi_{\mathcal{X}})$, so $\mathbf{x}^* = r\mathbf{1}$ and $\mathbf{y}^* = t\mathbf{1}$, where $r := 2\#(\mathbf{x})/K - 1 < 0$ and $t := 2\#(\mathbf{y})/K - 1 > 0$.

Finally, $\Gamma_{\mathcal{X}} \supseteq \Pi_{\mathcal{X}}$, so $\text{Fix}(\Gamma_{\mathcal{X}}) \subseteq \text{Fix}(\Pi_{\mathcal{X}}) \subseteq \mathbb{R}\mathbf{1}$. At this point, all hypotheses of Proposition 3.7 are verified; thus, $\mathcal{X}$ is McGarvey. $\square$

Proof of Example 3.9(b). Clearly $\Pi_{\mathcal{X}^{\text{eq}}_{\mathcal{N}}(r,R)} \supseteq \Pi_*$, so it is transitive. Thus, Corollary 3.8 says that $\mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)$ is McGarvey if and only if there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)$ with $\#(\mathbf{x}) < K/2 < \#(\mathbf{y})$.

Claim 1. *There always exists* $\mathbf{x} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)$ *with* $\#(\mathbf{x}) < K/2$.

Proof. Note that $N - r \geq 0$ because $r \leq R \leq N$. If $N - r$ is *even*, then let $L := \frac{N+2-r}{2}$ (≥ 1), and let $\mathbf{x} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}$ describe an equivalence relation where $\mathcal{N}$ splits into two equivalence classes of size L, along with $r - 2$ singleton classes. Then

$$\#(\mathbf{x}) = 2\left(\frac{L(L-1)}{2}\right) = L(L-1) < L\left(L - \frac{1}{2}\right) \underset{(*)}{\leq} \frac{N(N-1)}{4} = \frac{K}{2},$$

as desired. Here $(*)$ is because $L \leq N/2$ because $r \geq 2$.

If $N - r$ is *odd*, then $N - r \geq 1$. Let $L := \frac{N+1-r}{2}$ (≥ 1), and let $\mathbf{x} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}$ describe an equivalence relation where $\mathcal{N}$ splits into one equivalence class of size L, one class of size $L + 1$, and $r - 2$ singleton classes. Then

$$\#(\mathbf{x}) = \frac{L(L-1)}{2} + \frac{(L+1)L}{2} = \frac{2L^2}{2} = L^2 \underset{(*)}{<} \frac{N(N-1)}{4} = \frac{K}{2},$$

as desired. Here $(*)$ is because $L \leq (N-1)/2$ because $r \geq 2$.

In either the *even* or *odd* case, we have $\text{rank}(\mathbf{x}) = r \in [r \dots R]$ so $\mathbf{x} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)$. $\diamond$ claim 1

Claim 1 and Corollary 3.8 imply that $\mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)$ is McGarvey if and only if there exists $\mathbf{y} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)$ with $\#(\mathbf{y}) > K/2$. We must show this occurs if and only if $r < \overline{r}(N)$.

Let $M := N - r + 1$, and let $\mathcal{M} \subset \mathcal{N}$ be a subset of cardinality M, so that $|\mathcal{N} \setminus \mathcal{M}| = N - M = r - 1$. Let $\mathbf{y} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}$ describe the equivalence relation where $\mathcal{M}$ forms one equivalence class, and each element of $\mathcal{N} \setminus \mathcal{M}$ forms a singleton equivalence class, for r equivalence classes in total. Thus, $\text{rank}(\mathbf{y}) = r$, so $\mathbf{y} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)$. It is easy to see that $\#(\mathbf{y}) = \max\{\#(\mathbf{x}); \mathbf{x} \in \mathcal{X}^{\text{eq}}_{\mathcal{N}}(r, R)\}$. Thus, it suffices to show that $\#(\mathbf{y}) > K/2$ if and only if $r < \overline{r}(N)$. To see this, let

$$\overline{M} := N - \overline{r}(N) + 1 = \frac{1 + \sqrt{2N^2 - 2N + 1}}{2}.$$

Then $\overline{M}$ is the positive root of the polynomial $f(M) = M^2 - M - (N^2 - N)/2$. Thus, for any $M \in \mathbb{N}$, we have

$$(r < \overline{r}(N)) \iff (M > \overline{M}) \iff (f(M) > 0) \iff \left(M^2 - M > \frac{N^2 - N}{2}\right)$$
$$\underset{(*)}{\iff} \left(\frac{M(M-1)}{2} > \frac{K}{2}\right) \underset{(\dagger)}{\iff} \left(\#(\mathbf{y}) > \frac{K}{2}\right),$$

as claimed. Here, $(*)$ is because $K = N(N-1)/2$, and $(\dagger)$ is because $\#(\mathbf{y}) = M(M-1)/2$. $\square$

Proof of Proposition 4.2. "(a) $\Longrightarrow$ (c)" is clear.

"(c) $\Longrightarrow$ (b)" (by contrapositive) Let $\mathcal{O}_{-\mathbf{1}}$ be the open orthant containing $-\mathbf{1}$. If there is no $\mathbf{c} \in \operatorname{conv}(\mathcal{X})$ with $\mathbf{c} \ll \mathbf{0}$, then $\operatorname{conv}(\mathcal{X}) \cap \mathcal{O}_{-\mathbf{1}} = \emptyset$; thus, Theorem 1.3(a) says $-\mathbf{1} \notin \operatorname{maj}(\mathcal{X})$.

"(b) $\Longleftarrow$ (a)" If $\mathcal{X}$ is comprehensive, then $\operatorname{conv}(\mathcal{X})$ is also comprehensive. That is, for all $\mathbf{c} \in \operatorname{conv}(\mathcal{X})$ and $\mathbf{r} \in [-1, 1]^{\mathcal{K}}$, if $\mathbf{c} \le \mathbf{r}$, then $\mathbf{r} \in \operatorname{conv}(\mathcal{X})$ also. If $\mathbf{c} \in \operatorname{conv}(\mathcal{X})$ and $\mathbf{c} \ll \mathbf{0}$, then the set $\{\mathbf{r} \in [-1, 1]^{\mathcal{K}}; \mathbf{r} \gg \mathbf{c}\} \subseteq \operatorname{conv}(\mathcal{X})$ is an open neighbourhood of $\mathbf{0}$; thus, Theorem 1.3(b2) says $\mathcal{X}$ is McGarvey. □

Proof of Proposition 4.4. Proposition 1.1(b) says $\mathcal{X}$ is median-saturating if and only if $\mathcal{W}_2(\mathcal{X}) = \emptyset$. If $\mathcal{X}$ is comprehensive, then any $\mathcal{X}$-forbidden word must be all zeros. Thus, any element of $\mathcal{W}_2(\mathcal{X})$ has the form $(0_j, 0_k)$ for some $j, k \in \mathcal{K}$. Thus, $\mathcal{W}_2(\mathcal{X}) = \emptyset$ if and only if, for all $j, k \in \mathcal{K}$, there exists $\mathbf{x} \in \mathcal{X}$ with $x_j = 0 = x_k$. □

Proof of Proposition 5.1. First we must show that $\operatorname{span}(\mathcal{X}_f) = \mathbb{R}^{\mathcal{K}}$.

Claim 1. *If* $\operatorname{span}(\mathcal{X}_f) \neq \mathbb{R}^{\mathcal{K}}$, *then there is some* $j \in \mathcal{J}$ *and* $s_j \in \{\pm 1\}$ *such that* $f(\mathbf{x}) = s_j x_j$ *for all* $\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}$.

Proof. If $\operatorname{span}(\mathcal{X}_f) \neq \mathbb{R}^{\mathcal{K}}$, then for all $(\mathbf{x}, y) \in \mathcal{X}_f$, the coordinate y must be an affine function of $\mathbf{x}$; in other words, f must be an affine function. Thus, there are constants $s_j \in \mathbb{R}$ for all $j \in \mathcal{J}$, and another constant $r \in \mathbb{R}$ such that $f(\mathbf{x}) = r + \sum_{j \in \mathcal{J}} s_j x_j$ for all $\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}$.

Claim 1.1. *For all* $j \in \mathcal{J}$, *we have* $s_j \in \{-1, 0, 1\}$.

Proof. Let $\mathcal{I} := \mathcal{J} \setminus \{j\}$, Fix $\mathbf{x}_{\mathcal{I}} \in \{\pm 1\}^{\mathcal{I}}$. Then either $f(\mathbf{x}_{\mathcal{I}}, -1_j) = f(\mathbf{x}_{\mathcal{I}}, 1_j)$, or $f(\mathbf{x}_{\mathcal{I}}, -1_j) = -f(\mathbf{x}_{\mathcal{I}}, 1_j)$. But clearly,

$$f(\mathbf{x}_{\mathcal{I}}, 1_j) - f(\mathbf{x}_{\mathcal{I}}, -1_j) = r + \sum_{i \in \mathcal{I}} s_i x_i + s_j(+1) - r - \sum_{i \in \mathcal{I}} s_i x_i - s_j(-1) = 2s_j.$$

Thus, if $f(\mathbf{x}_{\mathcal{I}}, -1_j) = f(\mathbf{x}_{\mathcal{I}}, 1_j)$, then $s_j = 0$. If $f(\mathbf{x}_{\mathcal{I}}, -1_j) = -f(\mathbf{x}_{\mathcal{I}}, 1_j)$, then $s_j = \pm 1$. ▽ claim 1.1

Claim 1.2. *There is at most one* $j \in \mathcal{J}$ *such that* $s_j \neq 0$.

Proof (*By Contradiction*)**.** Suppose $s_j \neq 0 \neq s_k$ for some $j \neq k \in \mathcal{J}$. Let $\mathcal{I} := \mathcal{J} \setminus \{j, k\}$.
Fix $\mathbf{x}_{\mathcal{I}} \in \{\pm 1\}^{\mathcal{I}}$. If $s_j = s_k$, then $f(\mathbf{x}_{\mathcal{I}}, 1_j, 1_k) - f(\mathbf{x}_{\mathcal{I}}, -1_j, -1_k) = s_j(1 + 1 - (-1 - 1)) = 4s_j$, which is impossible because $f(\{\pm 1\}^{\mathcal{J}}) \subseteq \{\pm 1\}$ while $s_j = \pm 1$ (by Claim 1.1).
If $s_j = -s_k$, then $f(\mathbf{x}_{\mathcal{I}}, -1_j, 1_k) - f(\mathbf{x}_{\mathcal{I}}, 1_j, -1_k) = s_k(-(-1) + 1 - (-1 - 1)) = 4s_k$, which is again impossible because $f(\{\pm 1\}^{\mathcal{J}}) \subseteq \{\pm 1\}$ while $s_k = \pm 1$ (by Claim 1.1).
Either way, we have a contradiction. Thus, either $s_j = 0$ or $s_k = 0$. ▽ claim 1.2

Claim 1.2 implies that $f(\mathbf{x}) = s_j x_j + r$ for all $\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}$. Claim 1.1 says that $s_j = \pm 1$, while $f(\mathbf{x}) = \pm 1$ and $x_j = \pm 1$ by definition. Thus, $r = 0$; hence $f(\mathbf{x}) = s_j x_j$. ◇ claim 1

Thus, if $f(\mathbf{x})$ depends nontrivially on more than one coordinate of $\mathbf{x}$, then the conclusion of Claim 1 is contradicted; hence $\operatorname{span}(\mathcal{X}_f) = \mathbb{R}^{\mathcal{K}}$. Now,

$$\sum_{\mathbf{y} \in \mathcal{X}_f} \mathbf{y} = \sum_{\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}} (\mathbf{x}, f(\mathbf{x})) = (\mathbf{0}_{\mathcal{J}}, 0) = \mathbf{0}_{\mathcal{K}},$$

because $\sum_{\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}} f(\mathbf{x}) = 0$ by hypothesis, and clearly $\frac{1}{2^J} \sum_{\mathbf{x} \in \{\pm 1\}^{\mathcal{J}}} \mathbf{x} = \mathbf{0}_{\mathcal{J}}$. Thus, Theorem 1.3(b4) says $\mathcal{X}_f$ is McGarvey. □

Proof of Proposition 5.2. Let $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$; we want $\mu \in \Delta(\mathcal{X}_f)$ such that $\operatorname{maj}(\mu) = \mathbf{x}$. Recall $\mathcal{K} = \mathcal{J} \sqcup \{0\}$; write $\mathbf{x} = (\mathbf{x}_{\mathcal{J}}, x_0)$ for some $\mathbf{x}_{\mathcal{J}} \in \{\pm 1\}^{\mathcal{J}}$. Let $\mathcal{Y}_+ := f^{-1}\{1\}$ and $\mathcal{Y}_- := f^{-1}\{-1\}$; by hypothesis, both these spaces are McGarvey.

If $x_0 = 1$, then find some $\mu_{\mathcal{J}} \in \Delta(\mathcal{Y}_+)$ such that $\operatorname{maj}(\mu) = \mathbf{x}_{\mathcal{J}}$. Define $\mu \in \Delta(\mathcal{X})$ by $\mu(\mathbf{y}, 1) = \mu_{\mathcal{J}}(\mathbf{y})$ for all $\mathbf{y} \in \mathcal{Y}_+$. Then $\operatorname{maj}(\mu) = \mathbf{x}$. If $x_0 = -1$, then perform a similar construction using some $\mu_{\mathcal{J}} \in \Delta(\mathcal{Y}_-)$. □

Proof of Proposition 5.3. If f is monotone, then $f^{-1}\{1\}$ is a comprehensive subset of $\{\pm 1\}^{\mathcal{J}}$. Thus, hypothesis #1 and Proposition 4.2 imply that $f^{-1}\{1\}$ is McGarvey.

If f is monotone, then $-f^{-1}\{-1\}$ is also a comprehensive subset of $\{\pm 1\}^{\mathcal{J}}$. Thus, hypothesis #2 and Proposition 4.2 imply that $f^{-1}\{-1\}$ is McGarvey.

At this point, Proposition 5.2 implies that $\mathcal{X}_f$ is McGarvey. □

Proof of Proposition 6.2. (a) "$\Longrightarrow$" It suffices to show that, for any $j \in \mathcal{J}$, there is some $\mathcal{C}_j^* \in \mathfrak{C}$ such that $j \in \mathcal{C}_j^* \subseteq \mathcal{J}$; it follows that $\mathcal{J}$ is a union of $\mathfrak{C}$-elements.

Let $\mu \in \Delta^*(\mathcal{X}_{\mathfrak{C}})$ be such that $\operatorname{maj}(\mu) = \boldsymbol{\chi}^{\mathcal{J}}$. Let $j \in \mathcal{J}$. Then $\operatorname{maj}_j(\mu) = 1$, so $\widetilde{\mu}_j > 0$. Let $\mathfrak{C}_j := \{\mathcal{C} \in \mathfrak{C}; j \in \mathcal{C}\}$; then $\widetilde{\mu}_j = \sum_{\mathcal{C} \in \mathfrak{C}_j} \mu(\boldsymbol{\chi}^{\mathcal{C}}) - \sum_{\mathcal{C} \in \mathfrak{C} \setminus \mathfrak{C}_j} \mu(\boldsymbol{\chi}^{\mathcal{C}})$. Let $\mathcal{C}_j^* = \bigcap_{\mathcal{C} \in \mathfrak{C}_j} \mathcal{C}$; then $\mathcal{C}_j^* \in \mathfrak{C}$, and for all $k \in \mathcal{C}_j^*$, we have $\widetilde{\mu}_k \ge \sum_{\mathcal{C} \in \mathfrak{C}_j} \mu(\boldsymbol{\chi}^{\mathcal{C}}) - \sum_{\mathcal{C} \in \mathfrak{C} \setminus \mathfrak{C}_j} \mu(\boldsymbol{\chi}^{\mathcal{C}}) = \widetilde{\mu}_j > 0$; hence $\operatorname{maj}_k(\mu) = 1$, which means $k \in \mathcal{J}$. Thus, $\mathcal{C}_j^* \subseteq \mathcal{J}$, as claimed.

"$\Longleftarrow$" Let $\mathcal{C}_1, \ldots, \mathcal{C}_N \in \mathfrak{C}$, and let $\mathcal{J} := \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_N$; we will construct $\mu \in \Delta^*(\mathcal{X}_{\mathfrak{C}})$ such that $\operatorname{maj}(\mu) = \boldsymbol{\chi}^{\mathcal{J}}$. Define $\mu \in \Delta^*(\mathcal{X}_{\mathfrak{C}})$ as follows:

- Set $\mu[\mathbf{1}] := \frac{N-1}{2N-1}$.
- For all $n \in [1 \ldots N]$, set $\mu[\boldsymbol{\chi}^{\mathcal{C}_n}] := \frac{1}{2N-1}$.

Thus, for all $n \in [1 \ldots N]$ and $j \in \mathcal{C}_n$, we have $\widetilde{\mu}_j \geq 2(\frac{N-1}{2N-1} + \frac{1}{2N-1}) - 1 = \frac{1}{2N-1} > 0$, whereas for all $k \in \mathcal{K} \setminus \mathcal{J}$, we have $\widetilde{\mu}_j = 2(\frac{N-1}{2N-1}) - 1 = \frac{-1}{2N-1} < 0$. Thus, $\mathrm{maj}(\mu) = \boldsymbol{\chi}^{\mathcal{J}}$.

(b) "[i]$\Longrightarrow$[ii]" is immediate because Eq. (3) asserts $\mathrm{maj}(\mathcal{X}) \subseteq \mathrm{med}^{\infty}(\mathcal{X})$.

"[ii]$\Longrightarrow$[iii]" (by contrapositive) Let $k \in \mathcal{K}$, but suppose $\{k\} \notin \mathfrak{C}$. Define $\mathcal{C}_k^*$ as in part (a); then $k \in \mathcal{C}_k^*$ and $\mathcal{C}_k^*$ is the smallest element of $\mathfrak{C}$ which contains k. Now, $\mathcal{C}_k^* \neq \{k\}$, because $\{k\} \notin \mathfrak{C}$. Thus, there exists $j \in \mathcal{C}_k^* \setminus \{k\}$. Define the word $\mathbf{w} \in \{\pm 1\}^{\{k,j\}}$ by $w_k = 1$ and $w_j = -1$; then $\mathbf{w}$ is $\mathcal{X}_{\mathfrak{C}}$-forbidden. Thus, $\mathcal{W}_2(\mathcal{X}_{\mathfrak{C}}) \neq \emptyset$; thus, Proposition 1.1(b) implies that $\mathcal{X}_{\mathfrak{C}}$ is not median-saturating.

"[iii]$\Longrightarrow$[i]" follows immediately from part (a), because any subset of $\mathcal{K}$ can be written as a union of singleton sets. $\Box$

Proof of Theorem 7.1. "$S(\mathcal{X}) \leq 4(K+1)\sigma(\mathcal{X})$" Let $\mathcal{U} \subset \mathrm{conv}(\mathcal{X})$, and let $\epsilon > 0$. We say that $\mathcal{U}$ is ϵ-*dense* in $\mathrm{conv}(\mathcal{X})$ if, for all $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$, there exists some $\mathbf{u} \in \mathcal{U}$ with $\|\mathbf{u} - \mathbf{c}\|_\infty < \epsilon$.

Claim 1. *For any $M \in \mathbb{N}$, let $\mathcal{C}_M := \{\widetilde{\mu};\ \mu \in \Delta_M^*(\mathcal{X})\}$. Then $\mathcal{C}_M$ is a $(\frac{2(K+1)}{M})$-dense subset of* $\mathrm{conv}(\mathcal{X})$.

Proof. Let $\mathbb{Q}_M := \{\frac{n}{M};\ n \in \mathbb{N}\}$, and let $\mathbb{Q}_M^{\mathcal{X}}$ be the set of all functions $\mu : \mathcal{X} \longrightarrow \mathbb{Q}_M$ (thus, $\Delta_M^*(\mathcal{X}) \subset \mathbb{Q}_M$). For any $r \in \mathbb{R}_+$, we define $\lfloor r \rfloor_M := \frac{\lfloor Mr \rfloor}{M}$; this is the largest element of the set $\mathbb{Q}_M$ which is no greater than r. Note that $0 \leq r - \lfloor r \rfloor_M \leq 1/M$. Let $\mathbf{c} \in \mathrm{conv}(\mathcal{X})$; we must find some $\mu \in \Delta_M^*(\mathcal{X})$ such that $\|\widetilde{\mu} - \mathbf{c}\|_\infty < 2(K+1)/M$.

Carathéodory's theorem says there exists some subset $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| = K+1$, and some $\nu \in \Delta(\mathcal{Y})$, such that $\widetilde{\nu} = \mathbf{c}$. Now define $\lambda \in \mathbb{Q}_M^{\mathcal{Y}}$ by $\lambda(\mathbf{y}) := \lfloor \nu(\mathbf{y}) \rfloor_M$ for all $\mathbf{y} \in \mathcal{Y}$. Let

$$q := \sum_{\mathbf{y} \in \mathcal{Y}} |\nu(\mathbf{y}) - \lambda(\mathbf{y})| \leq \frac{|\mathcal{Y}|}{M} = \frac{K+1}{M}. \tag{14}$$

Then

$$\|\widetilde{\lambda} - \mathbf{c}\|_\infty = \|\widetilde{\lambda} - \widetilde{\nu}\|_\infty \leq q. \tag{15}$$

Observe that

$$\begin{aligned} 1 - \sum_{\mathbf{y} \in \mathcal{Y}} \lambda(\mathbf{y}) &= \sum_{\mathbf{y} \in \mathcal{Y}} \nu(\mathbf{y}) - \sum_{\mathbf{y} \in \mathcal{Y}} \lambda(\mathbf{y}) = \sum_{\mathbf{y} \in \mathcal{Y}} (\nu(\mathbf{y}) - \lambda(\mathbf{y})) \\ &= \sum_{\mathbf{y} \in \mathcal{Y}} |\nu(\mathbf{y}) - \lambda(\mathbf{y})| = q. \end{aligned} \tag{16}$$

Thus, $q \in \mathbb{Q}_M$ (because $\lambda \in \mathbb{Q}_M^{\mathcal{Y}}$). However, in general $q > 0$, so $\lambda \notin \Delta^*(\mathcal{X})$. Fix some $\mathbf{y}_0 \in \mathcal{Y}$, and define $\mu \in \Delta_M^*(\mathcal{X})$ as follows: $\mu(\mathbf{y}_0) := \lambda(\mathbf{y}_0) + q \in \mathbb{Q}_M$, and $\mu(\mathbf{y}) := \lambda(\mathbf{y})$ for all other $\mathbf{y} \in \mathcal{Y} \setminus \{\mathbf{y}_0\}$ (and of course $\mu(\mathbf{x}) := 0$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{Y}$). Then Eq. (16) implies that $\sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{Y}} \mu(\mathbf{y}) = 1$, so $\mu \in \Delta_M^*(\mathcal{X})$. Furthermore,

$$\|\widetilde{\mu} - \widetilde{\lambda}\|_\infty \leq |\mu(\mathbf{y}_0) - \lambda(\mathbf{y}_0)| = q. \tag{17}$$

Combining Eqs. (14), (15), and (17), we have $\|\widetilde{\mu} - \mathbf{c}\|_\infty \leq \|\widetilde{\mu} - \widetilde{\lambda}\|_\infty + \|\widetilde{\lambda} - \mathbf{c}\|_\infty \leq q + q \leq 2(K+1)/M$, as desired. $\Box$

Now, let $M := 4(K+1)\sigma(\mathcal{X})$; Then $\mathrm{conv}(\mathcal{X})$ contains the ball $\mathcal{B}(\frac{4(K+1)}{M})$. Given $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, let $\mathbf{x}' := \frac{2(K+1)}{M}\mathbf{x}$; then $\mathrm{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}}$ must contain the ball $\mathcal{B}' := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}};\ \|\mathbf{r} - \mathbf{x}'\|_\infty \leq \frac{2(K+1)}{M}\}$. But $\mathcal{C}_M$ is $(\frac{2(K+1)}{M})$-dense in $\mathrm{conv}(\mathcal{X})$ (by Claim 1), so $\mathcal{C}_M$ must intersect $\mathcal{B}'$. Thus, $\mathcal{C}_M$ intersects $\mathrm{conv}(\mathcal{X}) \cap \mathcal{O}_{\mathbf{x}}$; thus, there is some $\mu \in \Delta_M^*(\mathcal{X})$ with $\mathrm{maj}(\mu) = \mathbf{x}$.

"$\sigma(\mathcal{X}) \leq S(\mathcal{X})$" For every $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, there exists $N \leq S(\mathcal{X})$ and some $\mu^{\mathbf{x}} \in \Delta_N^*(\mathcal{X})$ such that $\mathrm{maj}(\mu^{\mathbf{x}}) = \mathbf{x}$. This means that $\widetilde{\mu}^{\mathbf{x}} \in \mathcal{O}_{\mathbf{x}}$. However, if $\mu \in \Delta_N^*(\mathcal{X})$, then every coordinate of $\widetilde{\mu}$ is an integer multiple of $1/N$. Thus, if $\widetilde{\mu} \in \mathcal{O}_{\mathbf{x}}$, then for all $k \in \mathcal{K}$ we have $\widetilde{\mu}_k \geq 1/N \geq 1/S(\mathcal{X})$ if $x_k = 1$, while $\widetilde{\mu}_k \leq -1/N \leq -1/S(\mathcal{X})$ if $x_k = -1$. Thus, if $\mathcal{C} = \mathrm{conv}\{\widetilde{\mu}^{\mathbf{x}};\ \mathbf{x} \in \{\pm 1\}^{\mathcal{K}}\}$, then $\mathcal{B}(\frac{1}{S(\mathcal{X})}) \subseteq \mathcal{C} \subseteq \mathrm{conv}(\mathcal{X})$. Thus, $S(\mathcal{X}) \geq \sigma(\mathcal{X})$. $\Box$

Proof of Proposition 7.2. (a) If $\mathcal{X}$ is McGarvey, then $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{X})]$. Thus, the boundary of $\mathrm{conv}(\mathcal{X})$ does not include $\mathbf{0}$. The boundary of $\mathrm{conv}(\mathcal{X})$ is a union of $(K-1)$-dimensional faces, each of which is a union of one or more simplices of the form $\mathrm{conv}(\mathbf{x}^1, \ldots, \mathbf{x}^K)$ for some $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \mathcal{X}$ (by Carathéodory's theorem).

Now, if $M := \lceil 1/\delta(\mathcal{X}) \rceil$, then $\frac{1}{M} \leq \delta(\mathcal{X})$. Thus, $\mathcal{B}(\frac{1}{M})$ is disjoint from every boundary simplex of $\mathcal{X}$. Thus, $\mathcal{B}(\frac{1}{M}) \subseteq \mathrm{conv}(\mathcal{X})$. Thus, $M \geq \sigma(\mathcal{X})$.

(b) Let $\delta := \delta(K)$. For all McGarvey $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$, we have

$$S(\mathcal{X}) \underset{(\dagger)}{\leq} 4(K+1)\sigma(\mathcal{X}) \underset{(@)}{\leq} 4(K+1)\lceil 1/\delta(\mathcal{X}) \rceil \underset{(*)}{\leq} 4(K+1)\lceil 1/\delta \rceil,$$

where $(\dagger)$ is by Theorem 7.1, (@) is by part (a), and $(*)$ is because $\delta(\mathcal{X}) \geq \delta$ for any $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ (by their definitions).

Now, find $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$ such that $\delta(\mathbf{x}^1, \ldots, \mathbf{x}^K) = \delta$, and let $\mathbf{y} \in \mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ be such that $\|\mathbf{y}\|_\infty = \delta$. Let $\mathbf{z} \in \{\pm 1\}^{\mathcal{K}}$ be such that $\mathbf{y} \in \mathcal{O}_{\mathbf{z}}$. Let $\mathcal{P} \subset \mathbb{R}^{\mathcal{K}}$ be the hyperplane containing $\mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$; then $\mathcal{P}$ cuts $\mathbb{R}^{\mathcal{K}}$ into two open half-spaces, $\mathcal{H}^+$ and $\mathcal{H}^-$, where $\mathbf{z} \in \mathcal{H}^+$ and $\mathbf{0} \in \mathcal{H}^-$. Let $\mathcal{X}' := \{\pm 1\}^{\mathcal{K}} \cap (\mathcal{H}^- \cup \mathcal{P})$. Then $\mathcal{X}'$ is McGarvey (because $\mathbf{0} \in \mathrm{int}[\mathrm{conv}(\mathcal{X}')]$). Also, $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \mathcal{X}'$, and $\mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ is one of the boundary faces of $\mathrm{conv}(\mathcal{X}')$ (because $\mathrm{conv}(\mathcal{X}') \subset \mathcal{H}^- \cup \mathcal{P}$). Thus, $\sigma(\mathcal{X}') \geq 1/\delta$ (because $\mathbf{y} \in \mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$). Thus $S(\mathcal{X}') \geq 1/\delta$, by Theorem 7.1.

(c) Without loss of generality, let $\mathcal{K} = [1 \ldots K]$. If $\mathbf{B} := [b_{jk}]_{j,k\in\mathcal{K}}$ is a $K \times K$ matrix, then let $\|\mathbf{B}\|_\infty := \max_{j,k\in\mathcal{K}} |b_{j,k}|$. We then define $\chi(K) := \max\{\|\mathbf{A}^{-1}\|_\infty;$ any invertible matrix $\mathbf{A} \in \{\pm 1\}^{K\times K}\}$. We will use a result of Alon and Vũ [2], which says that

$$\frac{K^{K/2}}{2^{2K+\mathcal{O}(K)}} \leq \chi(K) \leq \frac{K^{K/2}}{2^{K-1}}. \tag{18}$$

Left-hand inequality. Let $\mathbf{A} \in \{\pm 1\}^{K\times K}$ be such that $\|\mathbf{A}^{-1}\|_\infty = \chi(K)$. Let $\mathbf{B} := \mathbf{A}^{-1}$, and find $\ell, m \in [1 \ldots K]$ such that $|b_{\ell m}| = \chi(K)$. Let $\mathbb{R}_{\not-} := \{r \in \mathbb{R}; r \geq 0\}$.

Let $\mathbf{y} := \mathbf{B} \cdot \mathbf{1}$. For any $k \in [1 \ldots K]$, if $\mathbf{A}'$ is obtained by negating the kth row of $\mathbf{A}$, then $(\mathbf{A}')^{-1}$ is obtained by negating the kth column of $\mathbf{B}$, which in particular negates $b_{\ell k}$. By negating the rows of $\mathbf{A}$ and columns of $\mathbf{B}$ as required, we can assume that $b_{\ell k} \geq 0$ for all $k \in [1 \ldots K]$. Thus, $y_\ell = \sum_{k=1}^K b_{\ell k} \geq b_{\ell m} = \chi(K)$.

For any $k \in [1 \ldots K]$, if $\mathbf{A}'$ is obtained by negating the kth column of $\mathbf{A}$, then $(\mathbf{A}')^{-1}$ is obtained by negating the kth row of $\mathbf{B}$, and hence, the kth entry in $\mathbf{y}$. By negating the columns of $\mathbf{A}$ and rows of $\mathbf{B}$ as required, we can assume that $\mathbf{y} \in \mathbb{R}^K_{\not-}$. Thus, if $Y := \sum_{j=1}^K y_j$, then $Y \geq y_\ell \geq \chi(K)$.

Let $\mathbf{s} := \frac{1}{Y}\mathbf{y}$; then $\mathbf{s} \in \mathbb{R}^K_{\not-}$ and $\sum_{k=1}^K s_k = 1$. Let $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$ be the column vectors of $\mathbf{A}$; then $\mathbf{0} \notin \mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$, because $\mathbf{A}$ is invertible. Now, $\mathbf{As} = \sum_{k=1}^K s_k \mathbf{x}^k$, so $\mathbf{As} \in \mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$. However, $\mathbf{As} = \frac{1}{Y}\mathbf{1}$, so $\delta(\mathbf{x}^1, \ldots, \mathbf{x}^K) \leq \|\mathbf{As}\|_\infty = \frac{1}{Y}$. Thus,

$$\frac{1}{\delta(K)} \geq \frac{1}{\delta(\mathbf{x}^1, \ldots, \mathbf{x}^K)} \geq Y \geq \chi(K) \underset{(*)}{\geq} \frac{K^{K/2}}{2^{2K+\mathcal{O}(K)}},$$

where $(*)$ is by the left-hand Alon–Vũ inequality (18).

Right-hand inequality. Let $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$ be any points such that $\mathbf{0} \notin \mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$, and let $\delta := \delta(\mathbf{x}^1, \ldots, \mathbf{x}^K)$. Let $\mathbf{c} \in \mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ be such that $\|\mathbf{c}\|_\infty = \delta$, and let $\mathcal{Y} \subseteq \{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ be a minimal subset such that $\mathbf{c} \in \mathrm{conv}(\mathcal{Y})$.

Claim 1. *$\mathcal{Y}$ is linearly independent.*

Proof (*By Contradiction*)**.** Suppose $\mathcal{Y}$ is linearly dependent. Let $\mathcal{C} := \mathrm{conv}\{\mathcal{Y}\}$ and $\mathcal{V} := \mathrm{span}\{\mathcal{Y}\}$. Then $\mathbf{x} \in \mathcal{C} \subset \mathcal{V}$. Let $D := \dim(\mathcal{V})$ and $C := \dim(\mathcal{C})$; then $C \leq D$. Let $Y := |\mathcal{Y}|$; then $C \leq Y - 1$. If $\mathcal{Y}$ is linearly dependent, then $D \leq Y - 1$. There are now two cases:

- Suppose $C = D$. Then $\mathcal{C}$ has nonempty relative interior in $\mathcal{V}$, so the relative boundary of $\mathcal{C}$ in $\mathcal{V}$ is a union of faces of dimension $D - 1$. The point $\mathbf{c}$ lies on this relative boundary (because it minimizes $\|\bullet\|_\infty$); thus $\mathbf{c}$ lies in some $(D-1)$-dimensional face, so Carathéodory's theorem says $\mathbf{c} \in \mathrm{conv}(\mathcal{Z})$ for some $\mathcal{Z} \subseteq \mathcal{Y}$ with $|\mathcal{Z}| \leq D$. But $D = C \leq Y - 1$; thus, $\mathcal{Z}$ is a proper subset of $\mathcal{Y}$, contradicting the minimality of $\mathcal{Y}$.
- Suppose $C \leq D-1$. Carathéodory's theorem says $\mathbf{c} \in \mathrm{conv}(\mathcal{Z})$ for some $\mathcal{Z} \subseteq \mathcal{Y}$ with $|\mathcal{Z}| \leq C+1$. But $C+1 \leq D \leq Y-1$, so again $\mathcal{Z}$ is a proper subset of $\mathcal{Y}$, contradicting the minimality of $\mathcal{Y}$. $\diamond$ claim 1

By re-ordering if necessary, we can assume $\mathcal{Y} = \{\mathbf{x}^1, \ldots, \mathbf{x}^N\}$ for some $N \leq K$. Then, by replacing $\mathbf{x}^{N+1}, \ldots, \mathbf{x}^K$ with some other $\widetilde{\mathbf{x}}^{N+1}, \ldots, \widetilde{\mathbf{x}}^K \in \{\pm 1\}^{\mathcal{K}}$ if necessary, we can ensure that the set $\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$ is linearly independent. Let $\mathbf{A}$ be the $K \times K$ matrix whose columns are $\mathbf{x}^1, \ldots, \mathbf{x}^K$; then $\mathbf{A}$ is nonsingular. Let $\mathbf{B} := \mathbf{A}^{-1}$. Since $\mathbf{c} \in \mathrm{conv}\{\mathbf{x}^1, \ldots, \mathbf{x}^K\}$, we have $\mathbf{c} = \mathbf{As}$ for some $\mathbf{s} \in \mathbb{R}^K_{\not-}$ with $\sum_{k=1}^K s_k = 1$. Thus, $\mathbf{s} = \mathbf{Bc}$. Thus,

$$1 = \sum_{j=1}^K s_j = \sum_{j=1}^K \sum_{k=1}^K b_{jk} c_k,$$

where $\mathbf{c} = (c_1, \ldots, c_K)$. For all $k \in [1 \ldots K]$, we have $|c_k| \leq \|\mathbf{c}\|_\infty = \delta$. Thus,

$$1 = \left|\sum_{j=1}^K \sum_{k=1}^K b_{jk} c_k\right| \leq \sum_{j=1}^K \sum_{k=1}^K |b_{jk}||c_k| \leq \delta \sum_{j=1}^K \sum_{k=1}^K |b_{jk}| \leq \delta K^2 \cdot \chi(K).$$

Thus, $\displaystyle \frac{1}{\delta} \leq K^2 \cdot \chi(K) \underset{(*)}{\leq} \frac{K^{2+K/2}}{2^{K-1}}$, where $(*)$ is by the right-hand Alon–Vũ inequality (18). Since this holds for all $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \{\pm 1\}^{\mathcal{K}}$, we conclude that $\frac{1}{\delta(K)} \leq \frac{K^{2+K/2}}{2^{K-1}}$, as claimed. □

Proof of Proposition 7.3. (a) (Similar to the proof of Proposition 6.2(a) "$\Longleftarrow$") First we show $-\mathbf{1} \in \mathrm{maj}(\mathcal{X})$. Pick distinct $i, j, k \in \mathcal{K}$, and define $\mu \in \Delta_3(\mathcal{X})$ by $\mu[\boldsymbol{\chi}^i] = \mu[\boldsymbol{\chi}^j] = \mu[\boldsymbol{\chi}^k] = 1/3$; then $\widetilde{\mu}_\ell = -1/3$ or -1 for all $\ell \in \mathcal{K}$, so $\mathrm{maj}(\mu) = -\mathbf{1}$. Note that $3 \leq 2K - 3$ because $K \geq 3$.

Now let $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}} \setminus \{-\mathbf{1}\}$. Let $\mathcal{J} := \{j \in \mathcal{K}; x_j = 1\}$ and let $J := |\mathcal{J}|$ (hence $J \geq 1$, since $\mathbf{x} \neq -\mathbf{1}$). If $J = 1$ or K, then $\mathbf{x} = \boldsymbol{\chi}^k$ for some $k \in \mathcal{K}$ or $\mathbf{x} = \mathbf{1}$; hence $\mathbf{x} \in \mathcal{X}$ by hypothesis, and hence $\mathbf{x} \in \mathrm{maj}(\mathcal{X})$. Thus, we can assume that $2 \leq J \leq K - 1$. Define $\mu \in \Delta^*_{2J-1}(\mathcal{X})$ as follows:

- Set $\mu[\mathbf{1}] := \frac{J-1}{2J-1}$.
- For all $j \in \mathcal{J}$, set $\mu[\boldsymbol{\chi}^j] := \frac{1}{2J-1}$.

Thus, for all $j \in \mathcal{J}$ we have $\widetilde{\mu}_j = \frac{1}{2J-1}$, whereas for all $k \in \mathcal{K} \setminus \mathcal{J}$, we have $\widetilde{\mu}_j = \frac{-1}{2J-1}$. Thus, $\mathrm{maj}(\mu) = \mathbf{x}$. This works for any $\mathbf{x} \in \mathcal{X}$. Note that $2J - 1 \leq 2K - 3$ because $J \leq K - 1$. Thus, $S(\mathcal{X}) \leq 2K - 3$.

(b) Suppose without loss of generality that $\mathcal{K} = [1 \ldots K]$. For all $k \in \mathcal{K}$, let $\mathbf{e}^k := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the "1" appears in the kth coordinate. By hypothesis, there exist $\mathbf{x}^k, \mathbf{y}^k \in \mathcal{X}$ such that $x^k_k = 1 = y^k_k$, but $\mathbf{x}^k$ and $\mathbf{y}^k$ differ in every other coordinate. Thus, $\frac{1}{2}(\mathbf{x}^k + \mathbf{y}^k) = \mathbf{e}^k$.

Now, let $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ be arbitrary. Let $\mathcal{J} := \{j \in \mathcal{K}; x_j = 1\}$ and let $J := |\mathcal{J}|$. Define $\mu \in \Delta_{2J+1}(\mathcal{X})$ by

$$\mu := \frac{1}{2J+1} \left(\delta_{-\mathbf{1}} + \sum_{j \in \mathcal{J}} (\delta_{\mathbf{x}^j} + \delta_{\mathbf{y}^j}) \right).$$

(Here $\delta_{\mathbf{y}} \in \Delta^*(\mathcal{X})$ is the point mass at $\mathbf{y}$.) Thus, for all $j \in \mathcal{J}$, we have $\widetilde{\mu}_j = 2/(2J+1) - 1/(2J+1) = 1/(2J+1) > 0$. Meanwhile for all $k \in \mathcal{K} \setminus \mathcal{J}$, we have $\widetilde{\mu}_k = -1/(2J+1) < 0$. Thus, $\mathrm{maj}(\mu) = \mathbf{x}$, as desired.

(c) For all $k \in \mathcal{K}$, let $\mathbf{e}^k$ be as in part (b). By hypothesis, there exist $\mathbf{x}^k, \mathbf{y}^k \in \mathcal{X}$ such that $x^k_k \neq y^k_k$, but $\mathbf{x}^k$ and $\mathbf{y}^k$ agree in every other coordinate. Now $-\mathcal{X} = \mathcal{X}$, so $-\mathbf{y}^k \in \mathcal{X}$ also. Note that $x^k_k = -y^k_k$, and $\mathbf{x}^k$ and $-\mathbf{y}^k$ differ in every other coordinate. Thus, $\frac{1}{2}(\mathbf{x}^k - \mathbf{y}^k) = s_k \mathbf{e}^k$, for some $s_k \in \{\pm 1\}$. Likewise, $-\mathbf{x}^k \in \mathcal{X}$, and $\frac{1}{2}(\mathbf{y}^k - \mathbf{x}^k) = -s_k \mathbf{e}^k$.

Now, given any $\mathbf{z} \in \{\pm 1\}^{\mathcal{K}}$, define $\mu \in \Delta^*_{2K}(\mathcal{X})$ by:

$$\mu := \frac{1}{2K} \left(\sum_{\substack{k \in \mathcal{K} \\ z_k = s_k}} (\delta_{\mathbf{x}^k} + \delta_{-\mathbf{y}_k}) + \sum_{\substack{k \in \mathcal{K} \\ z_k = -s_k}} (\delta_{-\mathbf{x}^k} + \delta_{\mathbf{y}_k}) \right).$$

Thus, for every $k \in \mathcal{K}$, we have $\widetilde{\mu}_k = \frac{z_k}{K}$, so $\mathrm{maj}(\mu) = \mathbf{z}$, as desired. □

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